DESCRIPTIVE GEOMETRY

Millar
Maclin
Markwardt

DESCRIPTIVE GEOMETRY

BY

ADAM V. MILLAR
Assistant Professor of Drawing
University of Wisconsin

EDWARD S. MACLIN
Professor of Industrial Education
University of Tennessee

and

LORRAINE J. MARKWARDT
Engineer in Forest Products
U. S. Department of Agriculture

TRACY & KILGORE, PRINTERS
MADISON, WISCONSIN.
1919

PREFACE

In preparing this text in descriptive geometry, the authors have endeavored to present the subject in a simple, logical manner and in close conformity to commercial drafting practice, as well as to develop in the student the faculty of visualizing magnitudes in space.

One of the principal factors in accomplishing these desiderata is the omission of the ground line. When the views of several points are given without the ground line being shown, the distances of the points from the horizontal or vertical planes of projection are not determined. The front view, however, does show the **relative** heights of the points and the top view shows the **relative** distances of the points from the vertical plane. It is the **relative** distances of points of an object from a plane with which we are concerned, since the distance of the whole object from the plane of projection does not change the orthographic projection of the object on that plane.

When it becomes necessary to locate points which are at given distances from the planes of projection the ground line must of course be used. In all cases, however, even when the ground line is not shown on the drawing, it is understood to be at right angles to the line joining the top and front views of any point.

By the omission of the ground line, and therefore the traces of a plane, the student's attention is concentrated on the object or magnitude in space, and not on the planes of projection. This teaches the student to visualize the object rather than memorize the views of the object and leads to more original work. The subject is thus made easier because memorizing constructions and centering attention on the views of the object rather than on the object itself are the greatest obstacles which the student encounters in mastering the subject. Since the ground line is omitted in commercial work, the subject as presented in this text conforms to common practice.

While the *first* quadrant or angle is used in nearly all current text books on descriptive geometry, it is worthy of note that the *third* quadrant is used almost exclusively in the drafting offices of this country and at the various universities as well in the mechanical drawing courses which are a pre-requisite to descriptive geometry. It seems logical and advisable therefore to present the subject in the third quadrant as is done in this text. Since the elimination of the ground line dispenses with the necessity of using a particular horizontal or vertical plane, it is possible to represent objects such as cones in their natural positions rather than inverting them to bring their bases in a horizontal plane of projection. This removes the chief objection to the use of the third quadrant.

Another salient feature of this text is the extensive use of auxiliary views, that is views other than the top, front, or end views. These are taken from any direction which will aid in construction or give a clearer idea of the form of the object. These views train the visualizing powers of the student.

In beginning the study of descriptive geometry, students usually experience considerable difficulty in assuming, on their own initiative, satisfactory layouts for the given exercises and problems, and much valuable time is consumed in this preliminary work. The system of miniature layouts for black board work and the co-ordinate system of presenting problems for drafting room work are features of this text which go far toward relieving these difficulties and allowing the student to concentrate his attention on the solution of the problem.

Although the method used here of presenting the principles of descriptive geometry is new in American texts on the subject, it is used to some extent by French authors such as Javary, Pillet, "F. J.", and others. It has been used and developed at the University of Wisconsin for the past six years, with results which have been most satisfactory and gratifying.

The authors have consulted many descriptive geometries in preparing the following text, but are particularly indebted for suggestions and problems to the following: Javary, Pillet, "F. J.", MacCord, Smith, Church, Higbee, and Ferris.

CONTENTS

CHAPTER II

APPLICATIONS OF THE ELEMENTARY PRINCIPLES OF THE POINT, LINE, AND PLANE

CHAPTER III

CURVED LINES AND SURFACES

CHAPTER IV

PLANE SECTIONS AND DEVELOPMENTS OF CURVED SURFACES

CHAPTER V

INTERSECTIONS OF CURVED SURFACES

INTRODUCTION

It may help some instructors who contemplate using the following text-book to know how the authors have successfully used the book in their classes. With this in view, the following general method for conducting the course is suggested. Each instructor will, no doubt, need to alter the method to some extent to suit the conditions under which he works.

At the University of Wisconsin, descriptive geometry is given as a three credit course for one semester of eighteen weeks. Each week's work consists of one general lecture for all students in the course, one recitation and two two-hour drafting periods for each section. One of the two-hour drafting periods is sometimes turned into a one-hour recitation period.

At the lecture, the general principles involved in the next lesson are explained, general announcements made, and problems assigned for a home plate which is to be handed in at the beginning of the recitation period. At the recitation, the students are drilled in the analyses of the problems and then sent to the black board with some particular problem to solve. Black board problems are assigned from the miniature layouts which accompany many of the lists of problems in the text. This insures a satisfactory figure and requires less time than making layouts from co-ordinates.

In the drafting room, other and usually more difficult problems are assigned. The number solved in each drafting period varies from one to four in accordance with the difficulty of the problem. The co-ordinate system of presenting layouts for drafting room problems will be found very convenient. The student solves the problems and if he has time letters the statements of the problems. The work is done on a 11″ x 15″ sheet and is left in pencil. Neatness, clearness, and accuracy

are demanded. With the exception of a few plates, the work is completed and handed in at the close of each two-hour drafting period. The plates are then corrected, graded, and returned to the student at the next drafting period. This method of giving a plate to be completed each time the student comes to the drafting room has the following advantages: the student comes better prepared for his work, he wastes no time in the drafting room, and the attendance is improved.

Unannounced written quizzes are given in the drafting room about every three weeks.. Neatness and clearness of the construction counts for 15% of the grade.

DESCRIPTIVE GEOMETRY

FIRST PRINCIPLES

1. Representing an object. Some objects may be described by a written or oral statement in such a way that a clear idea of the size and form of the object may be obtained. For example, the statement "a cube having an edge 3″ long" gives a clear idea of the form and size of the object. But when the object becomes more complicated and it is necessary to show not only its form and size, but also its position in space or its position with reference to some other object, the description by means of an oral or written statement becomes very difficult. A picture or drawing of the object will usually give the desired information regarding it more quickly and acurately than any other form of expression. For this reason the draftsman makes use of working drawings to convey his ideas to the manufacturer.

It is the purpose of descriptive geometry to explain the methods employed in making such a drawing. Usually these principles are explained by reference to geometrical magnitudes, such as points, lines, and planes, but their application to material things can easily be made.

The study of descriptive geometry should enable a person to form a mental picture of an object of three dimensions from a drawing made on a plane sheet. This process is called visualizing the object. Visualizing not only enables a person to read a completed drawing, but it also helps the designer to form a mental picture of one part of an object and its position with respect to other parts even before a drawing of the object is made.

There are different methods of representing an object by means of a drawing, but all of the methods use the principle of projection as explained in the following article.

2. Projection. If from a point S, Fig. 1, straight lines are drawn through a series of points A, B, C, ... , the points a, b, c, ... in which these lines pierce the plane T, are the **projections** of the points A, B, C . . . . on this plane.

S is the point at which the eye is supposed to be located, and is called the **point of sight.**

A, B, C, ... are points such as corners of an object in space.

Sa, Sb, Sc . . . . are **lines of sight** or **projecting lines** of the points A, B, C . . . . .

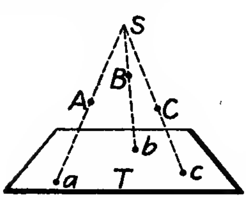


FIG. 1.—*Perspective.*

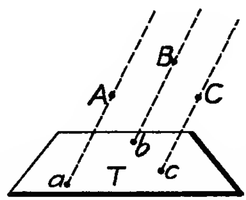


FIG. 2.—*Oblique projection.*

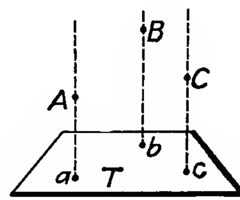


FIG. 3.—*Orthographic projection.*

T is the **picture plane** or the **plane of projection.**

Leaving color out of consideration, the projections a, b, c, ... present the same appearance to the eye, situated at the point of sight, as the points A, B, C, ... in space.

In **perspective,** Fig. 1, the point of sight is at a **finite distance** from the plane of projection. The projecting lines **diverge.** Such a view is a true picture of the object, similar to the view obtained with a camera.

In **oblique** and **orthographic** projection, Figs. 2 and 3, the point of sight is at an **infinite distance** from the plane of projection. The projecting lines are **parallel.**

The projection is **oblique** when the projecting lines are parallel to each other and oblique to the plane of projection. In perspective and oblique projections, the picture from one position is considered sufficient to represent the object.

The projection is **orthographic** when the projecting lines are perpendicular to the plane of projection. Since this form of projection is more commonly used than any other in engineering work, the greater part of this book will be devoted to the principles involved in orthographic projection.

CHAPTER I

THE ELEMENTARY PRINCIPLES OF THE POINT, STRAIGHT LINE, AND PLANE

3. A point in space is not completely determined by its orthographic projection on one plane, for the distance of the point from the plane is not shown by its projection. All points A, B, C, Fig. 4, which lie in a vertical straight line have the same projection on a horizontal plane. There are two methods of representing definitely a point in space. One method is to give its projection on a plane and also its distance from that plane. This method is used in making contour maps; a contour line being a line joining the projections of all points of an object which are a given distance above or below a base plane. The other method requires the use of two (or more) different planes of projection. When two projections are used, one is usually considered the principal projection. The other is supplementary, showing the distance of the point from the plane upon which the principal projection is made. The method in which two projections of an object are used is the more common.

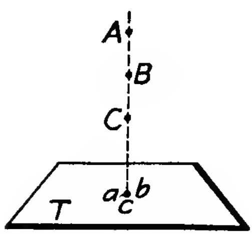


FIG. 4.—*a, b, c, projectons of points A, B, C, on plane T.*

4. Views or **projections of an object.** The **top view** of an object is the view obtained by looking directly down from the top. The point of sight is imagined to be an infinite distance directly above the object. The projecting lines for different points of the object are vertical and the plane of projection upon which this view or drawing is made is horizontal. This plane is called the **horizontal plane of projection or H.** The top view is also called the **horizontal projection** of the object and in architectural work it is called the **plan.**

The **front view** of an object is the view obtained by looking directly from the front. The projecting lines for this view are horizontal and are therefore at right angles to the projecting lines used in obtaining the top view. The plane upon which the front view is made is in a vertical position, and is called the **vertical plane of projection or V**. The front view is also called the **vertical projection** of the object, and in architectural work it is called the **front elevation.**

The horizontal and vertical planes of projection are at right angles to each other, Fig. 5. Their line of intersection is called the **ground line** or **G. L.** In most drawings, the object represented is considered to be below H and back of V. An object in this position is said to be in the **third quadrant or third angle.** In this case the planes of projection are between the points of sight and the object. In some drawings the object is considered to be above H and in front of V. In such a position the object is said to be in the **first quadrant** or **first angle.** Here the object is between the points of sight and the planes of projection.

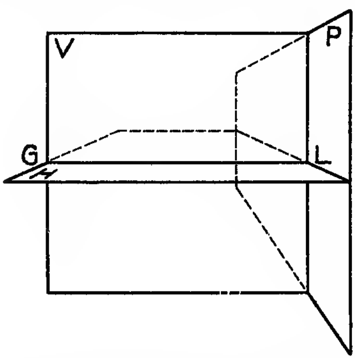


FIG. 5.—*Principal planes of projection.*

The **end or side view** of an object is the view obtained by lookiug directly from the left or right. The projecting lines for this view are parallel to the ground line. The plane upon which this view is made is perpendicular to the ground line and is called an **end** or **profile plane or P**. The end view is also called the **profile projection** and in architectural work the **end or side elevation.**

5. In this text, the term **auxiliary view** will be used for any other than the top, front, or end view. If the top and front views of an object are given, an auxiliary view taken by look-

ing **parallel** to H or V can be derived from these views. It must be remembered, however, that **the top and front views do not give information enough to construct easily an auxiliary view taken by looking obliquely to both H and V.** In Fig. 6, the right square pyramid is given by its top and front views. Three auxiliary views are shown, in each case the pro-

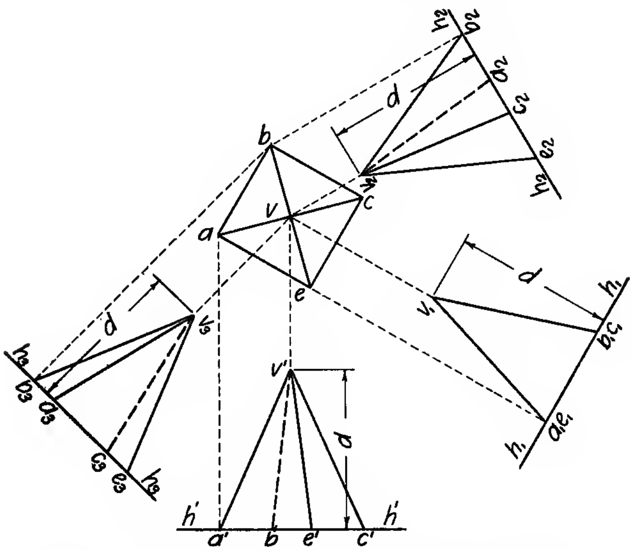


FIG. 6.—*Views of Pyramid. Orthographic Projection.*

jecting lines being parallel to H. In the front view, $h'h'$ represents a horizontal plane upon which the pyramid stands. This plane is represented in the different auxiliary views by h_1 h_1, h_2 h_2, and h_3 h_3. In each view, it is placed at right angles to the line of sight, but its distance from the top view is arbitrary. The altitude of the pyramid shows true length in all except the top view where it appears as a point. Notice that in different views, different edges are hidden and that none of the edges show true length in any of the views. In the first auxiliary view, the planes of two of the faces of the pyramid each appear as straight lines, one as the line a_1 v_1 and the other as b_1 v_1.

In the following pages, the terms top view, front view, end or side view, and auxiliary view will be used. It must not be forgotten that these terms are synonymous, respectively, to horizontal projection, vertical projection, profile projection, and auxiliary projection.

6. Relative position of views. In Fig. 7, the H, V, and P planes are shown in their proper relative positions. H and V are at right angles to each other, and P is at right angles to both of them, and therefore, at right angles to their line of intersection G. L. The point A is shown as being below H and back of V, that is, in the third quadrant. a is the top, a' the front, and a_1 the right end view of the point A. If all these views

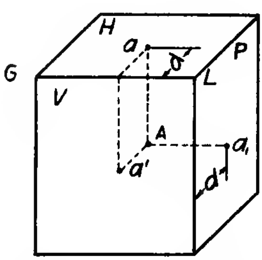


FIG. 7.—*Views of point A.*

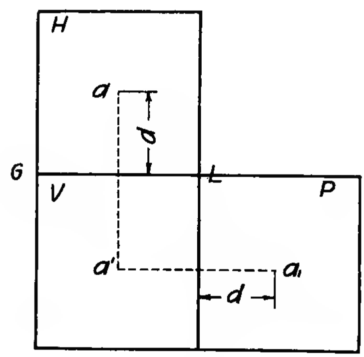


FIG.—8.

are shown on one sheet or drawing the different planes of projection H, V, P, etc., must all be considered as taken from their natural positions with respect to the object, and placed with the side which is turned away from the point of sight against the drawing surface. Notice that this was done in Fig. 6. The planes are usually placed with respect to each other as shown in Fig. 8. The ground line, G. L., is placed parallel to the edge of the T-square and the line joining a and a' at right angles to G. L. The end views are usually placed opposite the front view,

the right end view to the right of the front view as shown in Fig. 8 and the left end view, when shown, to the left of the front view. Sometimes it is more convenient to place the end views to the right and left of the top view.

If the object is in the first quadrant, that is above H and in front of V, the front view is placed above the top view on the drawing.

If the object represented is large or is drawn to a large scale, it frequently happens that each view is made on a separate sheet. This is usually the case in architectural work, a separate sheet being used for each floor plan, and still other sheets for front elevation, side elevation, etc.

POINTS

7. Views or projections of points. In Fig. 7, a is the top, a' the front, and a_1 the right end view of the point A. The point is below H and back of V, that is, in the third quadrant. When all these views are made on one sheet, they are usually placed with respect to each other as shown in Fig. 8. The distance from the top view of the point to the ground line is the same as the distance from the point in space to V. This distance also shows in the end view. The distance from the front view of the point to the ground line is the same as the distance from the point in space to H. When only one point in space is under consideration, the ground line must be shown on the drawing in order to fix definitely the position of the point; otherwise there wuold be nothing from which distances could be measured.

Fig. 9 represents two points A and B, and their positions with respect to H, V, and P. Fig. 10, shows the top, front, and end

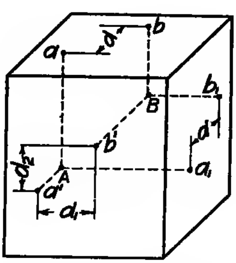


FIG. 9.—*Points A and B and their Views*

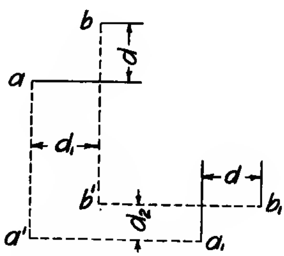


FIG. 10.—*Views of A and B*

views of the points A and B with the ground line omitted. The top view shows that B is the distance d farther back than A, and the front view shows that B is the distance d_2 higher than A. The end view shows both distances d and d_2. The top and front views both show that B is the distance d_1 to the right of A. If

two views of the points A and B are given, the third can be derived from these two. When the ground line is omitted, the drawing does not show how far the points, such as A and B, Fig. 10, are from H, V, and P, but it does give their true relative position. Thus the drawing, Fig. 10, shows that B is higher, farther back, and farther to the right than A.

It is the relative position of points of an object which determines the forms of its views or projections. As an illustration, suppose a three inch cube placed in a certain position and the top, front, and end views drawn. If each corner of the cube be lowered two inches, the top view of the cube will remain unchanged. The front and end views will remain the same in form, but may be placed lower on the sheet. But if one corner of the cube be placed one inch, another two inches, and a third three inches lower than they were originally, all views of the cube will be changed in form. The view of an object, therefore, does not depend upon the distances of its points from the picture plane, but rather upon the relative distances of its points from that plane.

Hereafter the ground line will rarely be shown on the drawing, but will be referred to at times to indicate direction. On the drawing, the ground line is at right angles to the line joining the top and front views of a point, and is usually parallel to the edge of the T-square. In space, it must be remembered as the line of intersection of H and V.

8. Picturing magnitudes in space from the drawing. In order to solve problems intelligently, the student should learn to picture to himself points, lines, and objects in their proper positions with reference to the drawing. When the drawing is in a horizontal position, it is best to consider the plane of the drawing as H, or the plane upon which the top view is made. The points of the magnitude should then be pictured as being directly above or below their top views. For example, in Fig. 10, the point B can be pictured as being on the plane of the drawing at b. Then A is the distance d_2 directly below its top view a. This gives a definite picture of the points A and B in space.

It is more difficult to picture the magnitude from its front view. To do this, hold the plane of the drawing, Fig. 10, in a vertical position. The point A can be pictured as coinciding with a' and B as located the distance d directly behind b'. This again gives a definite picture of the points A and B in space. While the plane of the drawing is in a vertical position, picture the point B directly behind b' the distance d. Now if the plane of the drawing be placed in its natural horizontal position, the point will be directly below b' the distance d. Therefore to picture magnitudes from their front views while the plane of the drawing is in a horizontal position, distance back must be thought of as distance down, while distance toward the front is distance up or above the front view.

One of the greatest benefits derived from the study of descriptive geometry is the ability to picture an object from the drawing. This is called "reading the drawing."

9. In the solution of a problem in descriptive geometry there are two parts, the analysis and the construction. The *analysis* of the problem states the general method of solution and refers usually to magnitudes in space and not to their views. The analysis for any problem is the same for all different positions of the magnitudes in space. The *construction* of the problem deals with the making of the drawing and explains how the lines are drawn to secure the required result. The construction changes with different positions of the magnitudes in space. The student should endeavor to keep these two parts of the problem as distinct as possible.

10. Letter notation. In descriptive geometry it is customary when referring to an object to designate points in space, such as corners, by capital letters, as A, B, C, etc. The top views or horizontal projections of these points are designated by lower case letters, as a, b, c, etc., while the front views are designated by lower case letters primed, as a', b', c', etc. If other than the top and front views of the object are shown, its corners are designated by lower case letters with subscripts, as a_1, b_1, c_1, etc., a_2, b_2, c_2, etc.

The lines which join two views of the same point are shown in the cuts of this book as fine dotted lines, but it is suggested that on pencil drawings these be made light full lines.

11. Problems. Solve in orthographic projection the following problems.

1. Given the top view of two points, A and B. Draw the front view of the points when A is ½″ higher than B.
2. Given the top and front views of two points. Draw the left end view of the points.
3. Given the front and right end views of two points. Draw the top view of the points.
4. Given the top and front views of three points. Draw the right end view of the points.
5. Given the front and left end views of three points. Draw the top view of the points.
6. Given the front view of two points A and B. Draw the top view of the points when A is 1″ farther back than B.
7. Draw the top, front, and right end views of two points A and B. B is 1″ higher, 2″ farther back, and 1½″ to the right of A.

STRAIGHT LINES

12. Views of a line. The word "line" will be taken to mean "straight line" unless statement to the contrary is made.

A line appears as a line in all views with the exception of the view taken by looking in the direction of the line itself; in this case it is a point.

At least two views of a line must be shown before the direction and length of the line are determined. The picture planes upon which these views are taken must not be parallel to each other. It is *not* sufficient to show just the top view or just the front view, but both top and front views must be shown, or one of these with some other such as the end view. If two views of a line are shown, other views can be derived from them.

Fig. 11 shows the top, front, and right end views of a line AB. The top view of the line is obtained by looking directly down from the top. This view shows that the line slopes back-

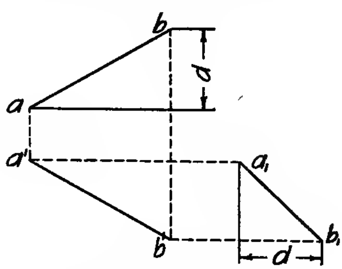


FIG. 11.—*Three Views of line A B.*

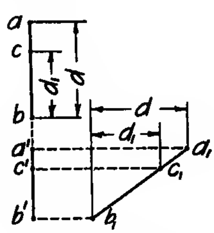


FIG. 12.—*Point C on line A B.*

ward to the right. The front view is obtained by looking directly from the front, and shows that the line slopes downward to the right. The right end view is obtained by looking from the right, in the direction of the ground line, and shows that the line slopes backward and downward. *d* is the distance which B is farther back than A. This distance shows in the top view and also in the end view.

If the top and front views of a line are given, the right end view can be found as follows: Since the front and end views both show height, draw lines through a' and b' parallel to G. L. At any distance to the right of the front view and on the line through a', select a point a_1. Through a_1 draw a line perpendicular to G. L. Take the distance d which b is back of a in the top view and set it off along the horizontal line through b′ to the right of the vertical line through a_1. This locates b_1. Then a_1 b_1 is the right end view of the line AB. Quite often in practical work, distances from front to back in the top view and from left to right in the end view are given from the center line of the object.

13. Views of lines which are in various positions with reference to H, V, and P.

If a line is oblique to H, V, and P, its top and front views are oblique to G. L.

In a manner similar to the above, describe the directions of the top and front views of lines in the following positions:

(a) Parallel to G. L.

(b) Parallel to H and oblique to V.

(c) Parallel to V and oblique to H.

(d) Perpendicular to H.

(e) Perpendicular to V.

(f) A line of profile. (A plane which is at right angles to the G. L. and cuts it at any point is a profile plane. A line of profile is any line which lies in a profile plane).

14. Point on line. If a point is on a line in space, the top view of the point is on the top view of the line, the front view of the point is on the front view of the line, etc., Fig. 13.

If the top and front views of a line AB are given, and also the top view of a point C on the line AB, the front view of C is found by drawing a perpendicular to G. L. through c and extending it to cut a' b' at c'. If the front view of C had been given the top view could have been located in a similar manner.

If the top and front views of a line are oblique to G. L., as AB, Fig. 13, the direction of the line is definitely determined

without having particular points of the line lettered or designated. If, however, the top and front views of a line are perpendicular to G. L., as they are with a line of profile AB, Fig. 12, both views of two points of the line must be lettered before the direction of the line is fixed. For example, suppose the top and front views of a line of profile are given as in Fig. 12, but having the front view unlettered. If the upper end of the front view is lettered a' and the lower end b', a definite line is represented. If, however, the lower end is lettered a' and the upper end b', a line of different direction is represented. This can be done since lines joining the top and front views of all points on a line of profile coincide. If the front view c' of a point on a given line of profile AB, Fig. 12, is represented, the top view of C cannot be found by the usual method of a perpendicular to G. L. The end view c_1 of the point is on the end view a_1 b_1, of the line. From the end view of the point, its top view c is located. If the top view of a point on the line had been given, the front view of the point could have been determined in like manner from the end view.

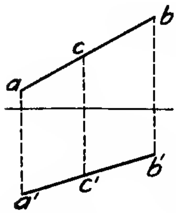


FIG. 13.—*Point C on line A B.*

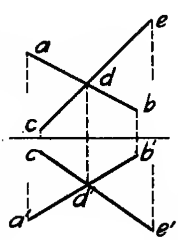


FIG. 14.—*Intersecting lines*

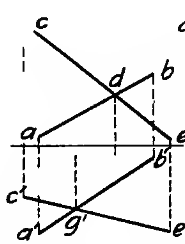


FIG. 15.—*Lines non-intersecting.*

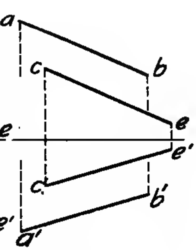


FIG. 16.—*Parallel lines.*

Intersecting lines. If two lines intersect in space, Fig. 14, the line joining the point of intersection d of their top views with the point of intersection d' of their front views must be perpendicular to G. L.

Non-intersecting lines. Two non-intersecting lines, Fig. 15, may be in such a position that their top views and also their

front views are intersecting. The line joining the point of intersection d of their top views with the point of intersection g' of their front views is not perpendicular to G. L.

Parallel lines. If two lines are parallel in space, Fig. 16, a view of the lines from any position, other than from a point on the plane of the lines, will show them as parallel lines. Hence any view of a parallelogram, from a point outside of the plane of the figure, is a parallelogram.

15. Coordinate system. In order to locate the problem in a convenient position on the sheet, the top and front views of the given points are determined by coordinates. Unless otherwise stated, the origin of coordinates is to be taken at the center of the space allotted to the problem. The base line is a horizontal line through the origin. The first coordinate gives the distance of the point to the right or left of the origin measured in the direction of the base line, to the right when positive, to the left when negative. The second coordinate locates the top view of the point, above the base line when positive, below when negative. The third coordinate locates the front view of the point, above the base line when positive, below when negative.

The problems for which coordinates are given in this text can be conveniently solved in a 5″ x 7″ rectangle unless statement to the contrary is made. The base line should be drawn through the center and parallel to the longer side of the rectangle. Distances are given in inches and should be used full scale unless otherwise stated. For blackboard work, multiply the coordinate by five. The cuts, accompanying lists of problems, are numbered to correspond with the problems to which they belong. It is suggested that they be used as layouts for blackboard work in the place of using the coordinates of the stated problem.

16. Problems.

1. Draw the left end view of the line joining the points A ($\frac{1}{2}$, $1\frac{1}{4}$, $-1\frac{3}{4}$) and B (2, $\frac{1}{4}$, $-\frac{3}{4}$).

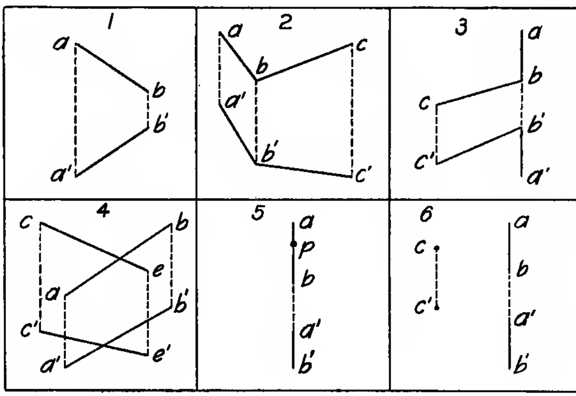
2. A ($-1\frac{1}{2}$, 1, $-\frac{1}{2}$), B ($-\frac{3}{4}$, 0, $-1\frac{3}{4}$), and C ($1\frac{1}{4}$, $\frac{3}{4}$, $-2\frac{1}{4}$) are three corners of a parallelogram. Draw its top and front views.

3. A $(-\frac{1}{4}, 1\frac{1}{2}, -1\frac{1}{2})$, B $(\frac{1}{4}, \frac{1}{2}, -\frac{1}{2})$, and C $(-2, 0, -1\frac{1}{4})$ are three corners of a parallelogram. Draw its right end view.

4. Given two lines A $(-\frac{3}{4}, 0, -2\frac{1}{4})$, B $(1\frac{1}{2}, 1\frac{3}{4}, -\frac{1}{2})$, and C $(-1\frac{1}{4}, 1\frac{3}{4}, -1\frac{1}{2})$, E $(1, \frac{3}{4}, -2\frac{1}{4})$. Do the lines intersect?

5. If P$(0, 1\frac{1}{4}, ?)$ lies on the line A$(0, 1\frac{3}{4}, -\frac{3}{4})$ B$(0, \frac{1}{4}, -2)$, locate its front view.

6. Locate the top and front views of a point E so that the line C $(-2, 1\frac{1}{4}, -\frac{1}{2})$ E (?, ?, ?) will be parallel to the line A $(-\frac{1}{2}, 1\frac{3}{4}, -1)$ B $(-\frac{1}{2}, \frac{1}{2}, -2\frac{1}{4})$.



REVOLUTION AND COUNTER-REVOLUTION

17. An object is said to revolve about a straight line as an axis when each of its points moves in the circumference of a circle whose center is in the axis and whose plane is perpendicular to the axis.

When an object is revolved about a straight line as an axis, the relative position of its points is not changed. The object can thus be brought into a simpler position with reference to the planes of projection. The views of the object in this position are easily found and from these the views of the object in its original position are located by the counter-revolution of its points.

If a point revolves about an axis, a view of its path taken by looking in the direction of the axis is a circle. A view of its path taking by looking in a direction perpendicular to the axis is a straight line at right angles to the axis and equal in length to the diameter of the circle.

If an axis of revolution is perpendicular to H, describe the top, front, and end views of the path of a point revolving about it. In a similar manner, describe the path of a point revolving about an axis which is perpendicular to V; one perpendicular to P.

18. To revolve a point about a horizontal axis.

Let P, Fig. 17, be the given point and SS the given axis.

An auxiliary view taken by looking in the direction of the axis SS, shows the path of the point P as a circle or the arc of a circle. The top view of this path is a straight line passing through p at right angles to ss. The line h_1h_1 in the auxiliary view represents a horizontal plane at the level of the axis SS. h_1h_1 is drawn at right angles to ss and at any convenient distance from the top view. The point s_1 on this line is the auxiliary view of the axis. In the front view, this horizontal plane is represented by the line $h'h'$. Since the auxiliary view is taken by looking in the direction of SS, the auxiliary view of P will be located on a line through p parallel to ss. The distance d is taken from the front view and used to locate p_1 in the auxiliary view, since height shows in both of these views. With s_1 as center and s_1p_1 as radius, the circle representing the path of the point in the auxiliary view is drawn. If the point is stopped at any position on the circle in the auxiliary view, its top and front views are found as follows: Through the auxiliary view of the

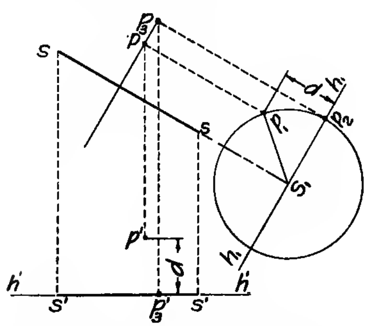


FIG. 17.—*P revolved about axis SS.*

point, draw a line parallel to ss until it intersects the top view of the circle, which is a line through p at right angles to ss. This intersection is the top view of the point. The front view of the point is directly below the top view, and the same distance above or below $h'h'$ that its auxiliary view is above or below h_1h_1. p_3 and p'_3 are the top and front views of the point when it is at the level of the axis SS. If the point makes a complete revolution about the axis, the auxiliary view of the path is a circle, the top view a straight line, and the front view an ellipse.

19. To revolve a point about an axis which is parallel to V.

Let P, Fig. 18, be the given point and FF the given axis.

An auxiliary view taken by looking in the direction of the axis FF, shows the path of the point P as a circle or the arc of a circle. The front view of this path is a straight line passing through p' at right angles to $f'f'$. The top view of the path is an ellipse or the arc of an ellipse. The line v_1v_1 in the auxiliary view represents a vertical plane containing the axis FF. v_1v_1 is drawn at right angles to $f'f'$ and at any convenient distance from the front view. The point f_1 on this line is the auxiliary view of the axis. In the top view, the vertical plane is represented by the line vv. Since the auxiliary view is taken by looking in the direction of FF, the auxiliary view of P will be located on a line through p' parallel to $f'f'$. The distance d is taken from the top view and used to locate p_1 in the auxiliary view, since distance from front to back is shown in both of these views. With f_1 as center and f_1p_1 as radius, the circle representing the path of the point in the auxiliary view is drawn. If the point is stopped at any position on the circle in the aux-

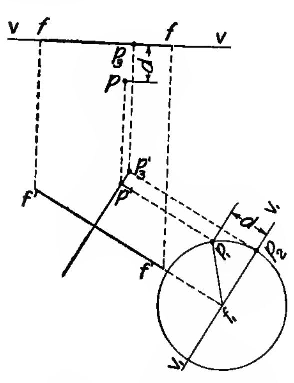


FIG. 18.—*P revolved about axis FF.*

iliary view, its top and front views are found as follows: Through the auxiliary view of the point, draw a line parallel to $f'f'$ until it intersects the front view of the circle which is a line through p' at right angles to $f'f'$. This intersection is the front view of the point. The top view of the point is directly above the front view and the same distance in front or back of vv that the auxiliary view of the point is from v_1v_1. p'_3 and p_3 are the front and top views of the point when it is the same distance from V as the axis FF.

If the axis of revolution is oblique to H and V, both the top and front views of the path of the point are ellipses. In this case the point cannot be revolved to a definite position in its path without the use of an auxiliary axis or two successive auxiliary views.

20. Problems.

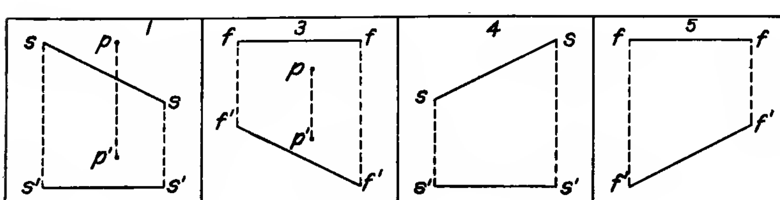
1. Rotate the point P $(-\frac{3}{4}, 1\frac{1}{2}, -\frac{7}{8})$ about S $(-2\frac{3}{8}, 1\frac{3}{4}, -1\frac{1}{2})$ S $(\frac{1}{4}, \frac{1}{4}, -1\frac{1}{2})$ as an axis until it reaches the level of SS. Letter the top and front views of the point in this position.

2. In problem 1, rotate P until it is $\frac{1}{2}''$ lower than SS and letter its top and front views in this position.

3. Rotate the point P $(1\frac{1}{4}, \frac{3}{8}, -1)$ through an angle of 60° about F $(-\frac{1}{4}, 1, -\frac{3}{4})$ F $(2\frac{3}{8}, 1, -2\frac{1}{4})$ as an axis. Letter the top and front views of the point in its final position.

4. S $(-\frac{1}{4}, \frac{1}{4}, -1\frac{1}{2})$ S $(2\frac{3}{8}, 1\frac{3}{4}, -1\frac{1}{2})$ is the center line of a shaft which carries a $1\frac{1}{2}''$ wheel at its center. If the wheel has eight spokes, letter the top and front views of the end of each spoke. (Use center lines only).

5. F $(-2\frac{3}{8}, 1, -2\frac{1}{4})$ F $(\frac{1}{4}, 1, -\frac{3}{4})$ is the center line of a shaft which carries a $1\frac{1}{2}''$ wheel at its mid point. If the wheel has eight spokes, letter the top and front views of the end of each spoke. (Use center lines only).



THE TRUE LENGTH OF A LINE

21. In orthographic projection, two views of a line will determine its direction in space, providing the plane upon which these views are taken are not parallel to each other or the line is not perpendicular to the line of intersection of the picture planes. If two points of the line are designated, the line will have a definite length between these points. Any view in which the line of sight is perpendicular to the given line will show the given line in its true length. In order to conveniently make an auxiliary view from two given views, the line of sight must be parallel to the plane upon which one of the given views is taken. Therefore to find the length of a line when two of its views are given, take an auxiliary view with the line of sight perpendicular to the given line and parallel to the plane of one of the given views.

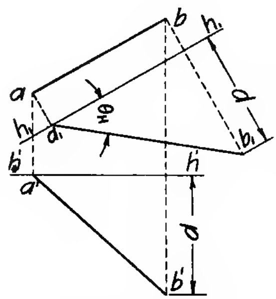
22. Given the top and front views of a line. Find the true length of the line and the angle which it makes with H.

Let AB, Fig. 19, be the given line.

Analysis. The angle which a line makes with H is the angle between the line and its top view. A view taken by looking at right angles to a plane containing the line and the top view will show the true length of the line and the angle which the line makes with H. In this case the line of sight will be parallel to H and perpendicular to the given line.

Construction. In Fig. 19, $h'h'$ is the front view of a horizontal plane which passes through A. Picture the line AB in space from its top view ab, A coinciding with a and B the distance d directly below b. Take an auxiliary view by looking at right angles to the plane abB. To do this draw lines at right angles to ab at the points a and b. At any point in the line through a, select the point a_1, and draw h_1h_1 through this point parallel to ab. h_1h_1 is the auxiliary view of the horizontal plane which has $h'h'$ for a

front view. b_1 is located on the line through b at right angles to ab and the distance d below h_1h_1. Then a_1b_1 is the true length of AB and θ_H the angle which AB makes with H.



• FIG. 19.—θ_H *angle AB makes with H.*

If AB had been parallel to V, the plane ab B would have been parallel to V, and the front view would have shown the true length of the line and its angle with H. If AB had been a line of profile, the end view would have shown its true length and its angle with H and V.

23. Given the top view ab of a line and the angle θ_H which AB makes with H, to find the front view of AB.

There are any number of lines which make 30° with H and have ab for a top view.

In Fig. 19, let ab be the given top view. At a and b, draw lines perpendicular to ab. At any point in the line through a, select the point a_1 and draw h_1h_1 through this point parallel to ab. Through a_1, draw the line a_1b_1 making the angle θ_H with h_1h_1. This gives the distance d which B is lower than A. Draw perpendiculars to G. L. through a and b. At any point on the perpendicular to G. L. through a, select the point a' and draw $h'h'$ through this point parallel to G. L. Take the distance d from the auxiliary view and set it below $h'h'$ locating the point b' on the perpendicular to G. L. through b. Then $a'b'$ is the required front view of AB.

If the top view and true length of AB had been given, the front view in this case could also have been found by the use of the auxiliary view. The point b_1 would have been located from the length of a_1b_1 rather than from the angle θ_H.

24. Given the top and front views of a line. Find the true length of the line and the angle which it makes with V.

Let AB, Fig. 20, be the given line.

Analysis. The angle which a line makes with V is the angle between the line and its front view. A view taken by looking at right angles to a plane containing the line and its front view will show the true length of the line and the angle which it makes with V. In this case the line of sight will be parallel to V and perpendicular to the given line.

Construction. In Fig. 20, vv is the top view of a vertical plane which passes through A. Picture the line AB from its front view $a'b'$, A coinciding with a' and B the distance d directly back of b'. If the drawing is in a horizontal position, distance back is pictured as distance below the front view. Take an auxiliary view by looking at right angles to the plane $a'b'$B. To do this draw lines at right angles to $a'b'$ at the points a' and b'. At any point in the line through a', select the point a_1 and draw v_1v_1 through this point parallel to $a'b'$. v_1v_1 is the auxiliary view of the vertical plane which has vv for a top view. b_1 is located on the line through b' at right angles to $a'b'$ and the distance d below v_1v_1. Then a_1b_1 is the true length of AB and θ_V the angle which AB makes with V.

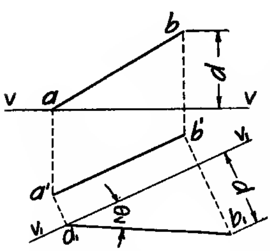


FIG. 20.—θ_V *angle AB makes with V.*

If AB had been parallel to H, the plane $a'b'$B would have been parallel to H, and the top view would have shown the true length of the line and its angle with V.

If the front view of the line is given and the angle which the line makes with V, the top view can be drawn as follows: From the front view and the angle which the line makes with V, draw the auxiliary view, Fig. 20. This view shows the distance d which one end of the line is further back than the other. With this distance, the top view can be determined.

There are any number of lines which make a given angle with V, and have a given front view.

If the front view and the true length of the line are given, the auxiliary view and then the top view can be found.

25. Second method. Given the top and front views of a line, find the true length of the line and the angle which it makes with H.

Let AB, Fig. 21, be the given line.

Analysis. Rotate the given line about an axis which cuts it and is perpendicular to H until the line is parallel to V. In this position, the front view will show the true length of the line and the angle which the line makes with H.

Construction. In Fig. 21, ac is the top and $a'c'$ the front view of an axis perpendicular to H. When AB is rotated about

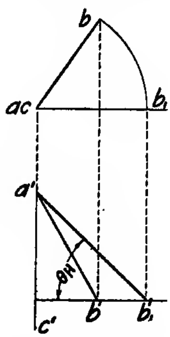


FIG. 21—θ_H *Angle AB makes with H.*

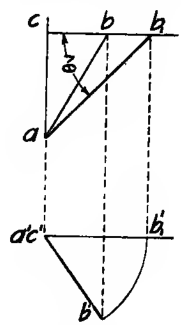


FIG. 22—θ_V *Angle AB makes with V.*

AC as an axis, A remains fixed and B moves in the arc of a circle to B_1, ab_1 being drawn parallel to G. L. The top view of the path in which B moves is the arc bb_1 and the front view the line $b'b_1'$ parallel to G. L. Since AB_1 is parallel to V, the front view $a'b'_1$ shows the true length of the line AB. θ_H is the true size of the angle which AB makes with H.

If the top view, ab, Fig. 21, of a line AB and the angle θ_H which the line makes with H are given, the front view of AB can be found as follows. With a as center and ab as radius, draw the arc bb_1; ab_1 being parallel to G. L. Draw perpendicu-

lars to G. L. through a and b_1. From any point as a' in the perpendicular through a, draw a line $a'b'_1$ making the angle θ_H with G. L. and cutting the perpendicular through b_1 at b'_1. When B_1 is revolved back to the position B, the front view b'_1 moves parallel to G. L. until it intersects at b' a line perpendicular to G. L. through b. Then $a'b'$ is the required front view.

26. Second method. Given the top and front views of a line, find the true length of the line and the angle which it makes with V.

Let AB, Fig. 22, be the given line.

Analysis. Rotate the given line about an axis which cuts it and is perpendicular to V until the line is parallel to H. In this position, the top view will show the true length of the line and the angle which the line makes with V.

Construction. In Fig. 22, $a'c'$ is the front and ac the top view of an axis perpendicular to V. When AB is rotated about AC as an axis, A remains fixed and B moves in the arc of a circle to B_1, $a'b'_1$ being drawn parallel to G. L. The front view of the path in which B moves is the arc $b'b'_1$, and the top view the line bb_1 parallel to G. L. Since AB_1 is parallel to H, the top view ab_1 shows the true length of line AB. θ_V is the true size of the angle which AB makes with V.

If the front view, $a'b'$, Fig. 22, of a line AB and the angle θ_V which the line makes with V are given, the top view of AB can be found as follows. With a' as center and $a'b'$ as radius, draw the arc $b'b'_1$; $a'b'_1$ being parallel to G. L. Draw perpendiculars to G. L. through a' and b'_1. From any point as a in the perpendicular through a', draw a line ab_1 making the angle θ_V with G. L. and cutting the perpendicular through b'_1 at b_1. When B_1 is revolved back to the position B, the top view b_1 moves parallel to G. L. until it intersects at b a line perpendicular to G. L. through b$'$. Then ab is the required top view.

27. Problems.

1. Find the true length of the line A($-\frac{3}{4}$, $\frac{1}{4}$, $-1\frac{1}{4}$) B($\frac{1}{2}$, $1\frac{1}{2}$, $-2\frac{1}{4}$) and the angle which it makes with H.

2. Find the true length of the line A($-\frac{3}{4}$, $\frac{3}{4}$, -2) B($\frac{1}{2}$, $1\frac{3}{4}$, $-\frac{3}{4}$) and the angle which it makes with V.

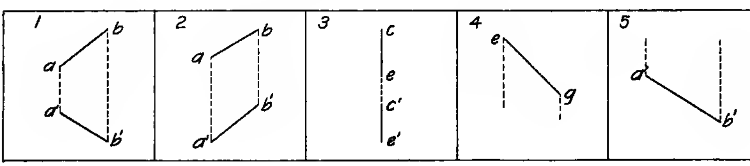
3. Find the true length of the line C(0, $1\frac{3}{4}$, $-\frac{1}{2}$) E(0, $\frac{1}{4}$, $-2\frac{1}{4}$) and the angle which it makes with H and V.

4. Draw the front view of the line E($-\frac{1}{2}$, $1\frac{3}{4}$, ?) G(1, $\frac{1}{4}$, ?) when its true length is $2\frac{1}{2}$″.

5. Draw the top view of the line A($-\frac{1}{2}$, ?, $-\frac{1}{4}$) B($1\frac{1}{2}$, ?, $-1\frac{1}{2}$) when the line makes 30° with V.

6. The three guy wires of a derrick run from the upper end of the mast O(0′, 12′, —6′) to points of support at A(—22′, 3′, —15′), B(14′, 34′, —23′), and C(24′, 2′, —19′). What is the maximum length of a boom fastened 3′ from the lower end of a 30′ mast which will clear all the guy wires? How long is each guy wire? (Scale $\frac{1}{8}$″ = 1′ —0″. Use a 7″ x 10″ rectangle for this problem).

7. A(—6′, 4′, —32′), B(—6′, 28′, —32′), C(18′, 28′, —32′), and E(18′, 4′, —32′) are the corners of the plate of a hip roof which has its peak at the point O(6′, 16′, ?). If the hip rafters make 30° with the horizontal, find their true lengths. Draw the front elevation of the roof. (Scale $\frac{1}{8}$″= 1′ —0″. Use a 7″ x 10″ rectangle for this problem.)



PLANES

28. A plane must usually be considered as indefinite in extent. Points or lines may be drawn on the plane, but do not bound it. In space, a plane is fixed or represented by three points, not in the same straight line, a point and a line, two intersecting lines, or two parallel lines. If the top and front views of these magnitudes are given, the plane which they determine will have its inclination with H and V definitely fixed.

The points A, B, and C, Fig. 23, represent a plane in space. It is evident that a line joining A and B, B and C, or A and C will lie in this plane.

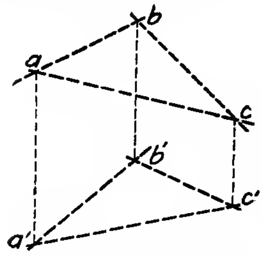


FIG. 23.—*Points A, B, C represent a plane.*

If a plane is represented by a point and a line, other lines of a plane can be found by joining the given point with points on the given line.

If the plane is represented by two lines, either parallel or intersecting, any number of other lines of the plane can be found by joining a point in one of the given lines with a point in the other. When the lines are intersecting, Fig. 14, the line which joins the top and front views of their point of intersection must be at right angles to the ground line. If the lines are parallel, Fig. 16, all views of the lines will show them parallel. If the lines are neither parallel nor intersecting, Fig. 15, they do not represent a plane.

If a plane is perpendicular to H, all lines of the plane which are of indefinite length, have the same top view. The front views of these lines will be either parallel or intersecting lines.

If a plane is perpendicular to V, all lines of the plane which are of indefinite length have the same front view. The top views of these lines will be either parallel or intersecting lines.

If a plane is perpendicular to both H and V, all lines of the plane which are of indefinite length have the same top view and also the same front view.

If one line of a plane is parallel to G. L., the plane is parallel to G. L. *A line is parallel to G. L. when both its top and front views are parallel to G. L.* This line with any point outside the line will represent a plane parallel to G. L.

29. Lines on the plane. The two intersecting lines MN and OP, Fig. 24, represent a plane. The line AB, joining the point A in MN with the point B in OP, is another line of the plane. *ab* is its top and *a′b′* its front view.

A line which lies in a given plane and is parallel to H, is called a horizontal line of that plane. In Fig. 25, SS is such a line which lies in the plane of the lines MN and OP. Its front view *s′s′* is drawn first and then its top view *ss* is found by locating the top views of the points X and Y where it crosses the given lines MN and OP. A line which lies in a given plane and is parallel to V is called a frontal line of that plane. FF, Fig. 26 is such a line which lies in the plane of MN and OP. Its top view *ff* is drawn first and then its front view *f′f′* is found. These lines which lie in a plane and are parallel to H or V, play an important part in the solution of many problems.

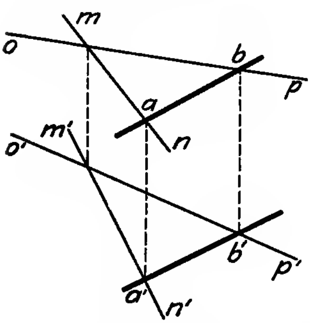


FIG. 24.—*AB line in plane of MN and OP.*

30. Points on planes. A point of a plane can be represented by placing the views of the point on the views of some line of the plane. If the top view of the point P which lies on the plane is given, the front view of P can be found as follows: Draw the top view of any line through the top view of P. Find the front view of the line by finding the front view of the points where it crosses the lines representing the plane. Then the front

view of P is on the front view of this line. If the front view of P had been given, the top view could have been found in a similar manner.

All points of the plane which are a given distance, say 1″ higher or lower than any point P, lie on a horizontal line of the plane. The true distance which the point is higher or lower than P is shown in the front view. Likewise, all points of the plane

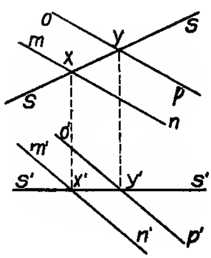


Fig. 25.—*SS horizontal line of plane.*

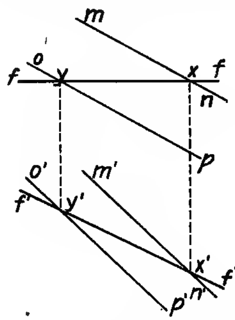


Fig. 26.—*FF frontal line of plane.*

which are a giving distance farther back or farther toward the front than a point P lie on a frontal line of the plane. The true distance which the point is farther to the front or back than P is shown in the top view.

31. Problems. In the following problems the given planes may be represented by the top and front views of any two of their lines unless otherwise stated.

1. Having given the top view of a line which lies in a given plane, to find its front view.

2. Having given the front view of a line which lies in a given plane, to find its top view.

3. Having given the top view of a point which lies in a given plane, to find its front view.

4. Having given the front view of a point which lies in a given plane, to find its top view.

5. A point and two parallel lines are given by their top and front views. Is the point in the plane of the two parallel lines?

6. Having given a plane which is parallel to the ground line and the top view of a point of this plane, find the front view of the point.

7. Having given a plane which is parallel to the ground line, and the top view of a line of this plane, find the front view of the line.

8. Having given a plane which is parallel to G. L., and the front view of a line of this plane which is also parallel to G. L, find the top view of the line.

9. Two lines are in such a position that neither their top nor their front views intersect within the limits of the drawing. Determine graphically whether or not the lines intersect.

10. A plane is represented by two intersecting lines. The top view of a line which passes through their point of intersection and lies in the given plane is given. Find the front view of the line.

11. A plane is represented by two parallel lines. The top view of another line of this plane which is parallel to the first two lines is given. Find the front view of the line.

12. Determine graphically whether or not the lines A ($-1\frac{1}{4}$, $1\frac{1}{2}$, $-2\frac{1}{4}$) B ($\frac{3}{4}$, $1\frac{3}{4}$, -2) and C ($-\frac{3}{4}$, $\frac{1}{8}$, $-\frac{3}{4}$) E ($1\frac{1}{4}$, $\frac{5}{8}$, $-\frac{3}{4}$) intersect. Do not prolong the lines AB and CE.

13. Draw the top and front views of a horizontal line and a frontal line on the plane M ($-1\frac{1}{2}$, $\frac{1}{4}$, $-1\frac{3}{4}$), N (1, $1\frac{3}{4}$, $-\frac{3}{4}$), O (1, $\frac{1}{2}$, $-2\frac{1}{4}$).

14. Find the top and front views of the locus of all points which lie on a given oblique plane ABC and are 1″ lower than A.

15. Find the top and front views of the locus of all points which lie on a given oblique plane ABC and are $1\frac{1}{4}$″ farther back than B.

16. Find the top and front views of a point which lies on a given oblique plane ABC and is $\frac{3}{4}$″ higher than A and 2″ farther back than B.

32. A point and two lines are given by their top and front views. Represent a plane which contains the given point and is parallel to the given lines.

Analysis. Through the given point draw a line parallel to each of the given lines. The plane of these two lines is the required plane.

Let the construction be made in accordance with the above analysis.

33. Problems.

1. Represent a plane which contains a given line and is parallel to another given line.

2. Represent a plane which contains a given point and is parallel to G. L.

3. Two lines parallel to the G. L. are given by their top and front views. Represent a plane which contains a given point and is parallel to the given lines.

4. Two lines, one oblique to G. L. and the other a line of profile, are given by their views. Represent a plane which contains a given point and is parallel to the given lines.

5. Represent a plane which contains a given point and passes at equal distances from two other given points. (Two solutions).

6. Represent a plane which contains a given line and passes at equal distance from two given points. (Two solutions.)

7. Represent a plane which contains a given point and passes at equal distances from three other given points. (Four solutions).

34. To find the angle which a given oblique plane makes with a horizontal plane.

Fig. 27 represents a triangular prism resting on one of its lateral faces.

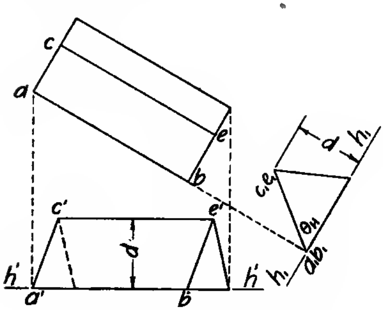


FIG. 27.—*Angle plane makes with H.*

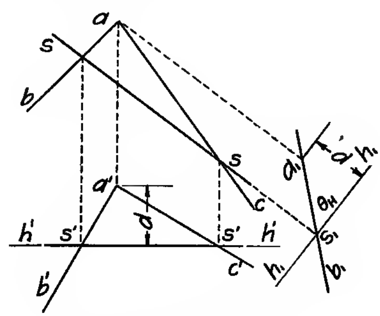


FIG. 28.—*Angle plane ABC makes with H.*

Let the given oblique plane be the face ABCE of the prism, and let the face upon which the prism rests be the horizontal plane.

Analysis. A view taken by looking in the direction of the line of intersection of the planes will show each plane as a straight line. The angle between these lines is the angle between the planes.

Construction. ab is the top and $a'b'$ the front view of the lateral edge of the prism, which is the line of intersection of the given planes. In the auxiliary view, h_1h_1 which is at right angles to ab, represents the horizontal plane upon which the prism rests. $h'h'$ is the front view of this plane. The distance d which the edge CE is above the plane of the opposite face is

taken from the front view and used to locate c_1e_1 in the auxiliary view. a_1c_1 represents the plane of the face of the prism in the auxiliary view and θ_H is the angle which this face makes with the horizontal plane upon which the prism rests.

35. To find the angle which a given oblique plane makes with a horizontal plane when the oblique plane is represented by two intersecting lines AB and AC. Fig. 28.

In the front view the horizontal plane is represented by one line $h'h'$ parallel to G. L. The top view of this horizontal plane is unlimited in extent, covering all of H, and therefore need not be represented by any straight lines. The intersection of this horizontal plane with the plane ABC is the horizontal line SS. In the auxiliary view taken by looking in the direction of the line SS, h_1h_1, which is at right angles to ss, represents the horizontal plane. The distance d taken from the front view, locates a_1 in the auxiliary view. a_1b_1 represents the oblique plane ABC in the auxiliary view and θ_H the angle which it makes with the horizontal plane.

36. Conversely, to represent a plane which makes a given angle with H.

There are any number of planes which make a given angle, say 30°, with H. Any horizontal line SS, Fig. 29, can be drawn and a plane represented which contains this line and makes 30° with H. In the auxiliary view taken by looking in the direction of SS, h_1h_1 at right angles to ss, represents H and s_1a_1, at 30° with h_1h_1, represents the required plane. The top view of the point A can be considered to be any point on the line ab, parallel to ss. The front view of AB is located by taking the distance d from the auxiliary view and setting it above $h'h'$ in the front view, $a'b'$ being parallel to $s's'$. The required plane is repre-

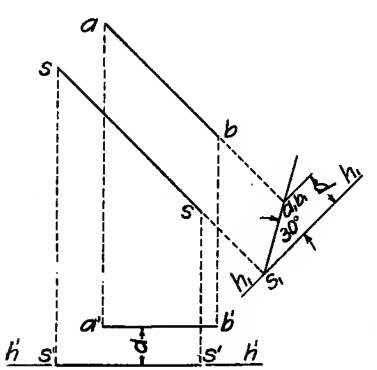


FIG. 29.—*Plane 30° with H.*

sented by the two parallel lines SS and AB or by SS and any point on the line AB. Other oblique lines of the plane can be drawn by joining points on AB with points on SS.

There are two planes which contain SS and make 30° with H. The other one would be represented in the auxiliary view by a line passing through a_1, sloping downward and making 30° with h_1h_1.

37. To find the angle which a given oblique plane makes with a plane parallel with V.

Let ABC, Fig. 30, represent the oblique plane and vv the top view of the plane which is parallel to V.

Analysis. A view taken by looking in the direction of the line of intersection of the planes will show each plane as a straight line. The angle between these lines is the angle between the planes.

Construction. ff is the top view and $f'f'$ the front view of the line of intersection of the planes. In the auxiliary view, v_1v_1 which is at right angles to $f'f'$, represents the plane which is parallel to V. The distance d which the point A is back of this plane is taken from the top view and used to locate a_1 in the auxiliary view. a_1f_1 represents the oblique plane in the auxiliary view and θ_V is the angle which this plane makes with the plane parallel with V. The oblique plane also makes this same angle θ_V with V.

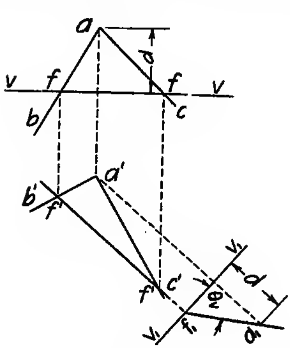


FIG. 30.—θ_V *angle plane makes with V.*

In the auxiliary view, a_1 is placed nearer the front view than the line v_1v_1 when the point A is in front of the vertical plane as determined by the top view. If A is back of the vertical plane then a_1 is placed on the other side of v_1v_1.

38. Conversely, to represent a plane which makes a given angle with V.

There are any number of planes which make a given angle, say 45°, with V. Any frontal line FF, Fig. 31, can be drawn and a plane represented which contains this line and makes 45° with V. In the auxiliary view taken by looking in the direction of FF, v_1v_1 at right angles to $f'f'$, represents V and f_1a_1, 45° with v_1v_1, represents the required plane. Each point of the line f_1a_1 represents a line which is parallel to FF and lies in the required plane. a_1b_1 represents such a line. $a'b'$ parallel to $f'f'$ is its front view. The top view of AB is located by taking the distance d from the auxiliary view and placing it back of vv in the top view, ab being parallel to ff.

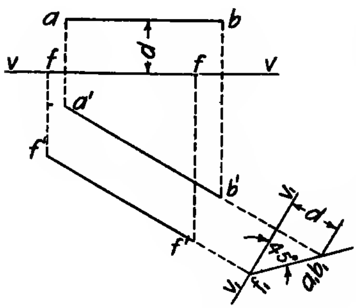


FIG. 31.—*Plane 45° with V.*

The required plane is represented by the two parallel lines FF and AB, or by FF and any point on the line AB. Other oblique lines of the plane can be drawn by joining points on AB with points on FF.

There are two planes which contain FF and make 45° with V. The other one slopes forward making 90° with the plane of FF and AB.

39. Problems.

1. Find the angle which the plane A(−2½, ¾, −2¼) B(−1, 1½, −¾) C(½, ¼, −1¾) makes with H.

2. A(−2¾, 1⅝, −4) B(−1⅝, 3⅛, −4) C(−⅛, 2, −4) E(−1¼, ½, −4) is the base of a regular square pyramid whose altitude is 2½". If V is the vertex, find the angle which the face VBC makes with the plane of the base. (Use a 7"x10" rectangle for this problem.)

3. Vertical bore holes are driven to a certain vein of ore at three points A(−550, 400, ?) B(−250, 700, ?) and C(50, 200, ?). The bore holes strike the vein at 650', 1050' and 750' above sea level, respectively. If the direction of the G. L. is considered to be East and West, find the strike and dip of the vein. Scale 1" = 200'. (Use a 7" x 10" rectangle for this problem.)

NOTE. By strike is meant a horizontal line on the plane of one wall of the ore body. The direction of the strike is the same as the direction of outcrop of the vein along a horizontal surface. By dip is meant the angle of inclination of the wall with a horizontal surface.

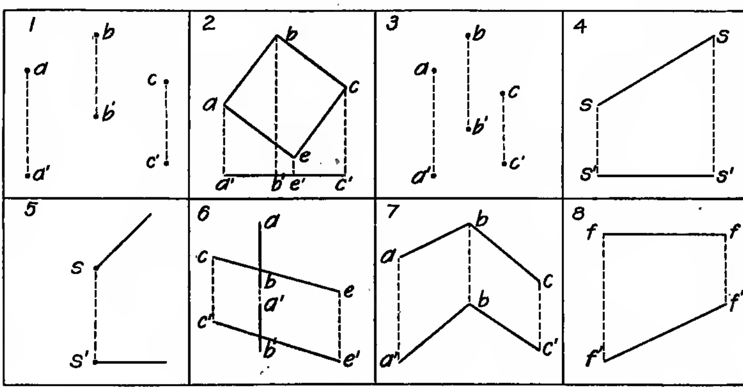
4. S($-\frac{1}{4}$, $\frac{1}{4}$, $-1\frac{1}{2}$) S($2\frac{3}{8}$, $1\frac{3}{4}$, $-1\frac{1}{2}$) is a horizontal line of a plane which makes 30° with H. Represent the plane in the top and front views.

5. The strike of a certain ore vein is N 45° E from the point S(0, $\frac{1}{4}$, $-1\frac{1}{2}$). Represent one wall of the vein when the dip is 45° to the NW.

6. Find the angle which the plane of A(1, $3\frac{1}{2}$, -2) B(1, $1\frac{1}{4}$, $-3\frac{1}{2}$) and C(0, $2\frac{1}{2}$, $-2\frac{1}{2}$) E($2\frac{3}{4}$, $1\frac{1}{8}$, $-3\frac{7}{8}$) makes with H and V. (Use a 7″ x 10″ rectangle for this problem.)

7. Find the angle which the plane A($-2\frac{3}{4}$, $\frac{3}{4}$, $-2\frac{1}{4}$) B ($-1\frac{1}{4}$, $1\frac{1}{2}$, $-\frac{3}{4}$) C($\frac{1}{4}$, $\frac{1}{4}$, $-1\frac{3}{4}$) makes with V.

8. Represent a plane which contains the line F($-2\frac{3}{8}$, 1, $-2\frac{1}{4}$) F($\frac{1}{4}$, 1, $-\frac{3}{4}$) and makes 45° with V.



PLANE FIGURES

40. If two views of the magnitudes which represent a plane are given, the position of the plane is determined. These magnitudes may form some geometrical figure as an angle, a triangle, a square, etc., or such a geometrical figure may be on the plane in addition to the magnitudes which represent the plane. A view of such a plane figure taken with the line of sight perpendicular to the plane will show the figure in its true size and and shape. Consequently, if the plane of the figure is parallel to H, V, or P, the top, front, or end view respectively will show the true size of the figure. If the plane of the figure is oblique to both H and V, the true size of the figure can be found by rotating the plane, and the figure also, until it is parallel to H when the top view will show the true size, or else parallel to V, when the front view will show its true size.

In order to rotate the plane of a figure until it is parallel to some other plane, the axis of rotation must be parallel to the line of intersection of the two planes.

Another method for finding the true size of a figure when the plane of the figure is oblique to H and V, is to take two auxiliary views of it. The line of sight for the first auxiliary view must be parallel to the picture plane of one of the given views and also from such a direction that the plane of the figure appears as a line. The second auxiliary view is derived from the first and has the line of sight perpendicular to the plane of the figure, thus showing the true size of the figure.

The second method is very convenient at times, but students frequently find it more difficult than the method of revolution. It is therefore not recommended to students at this stage of the subject.

41. Two intersecting lines are represented by their top and front views. Find the true size of the angle between them.

Let AO and BO, Fig. 32, be the given lines.

Analysis. Draw a horizontal line in the plane of the given lines. Revolve the given lines about the horizontal line as an axis until the plane of the lines is parallel to H. The lines do not change their relative position during the revolution. The top view of the angle in the revolved position shows the true size of the angle between the lines.

Construction. $s's'$ is the front and ss the top view of a horizontal line on the plane of AO and BO. In the auxiliary view taken by looking in the direction of the axis, h_1h_1 represents a

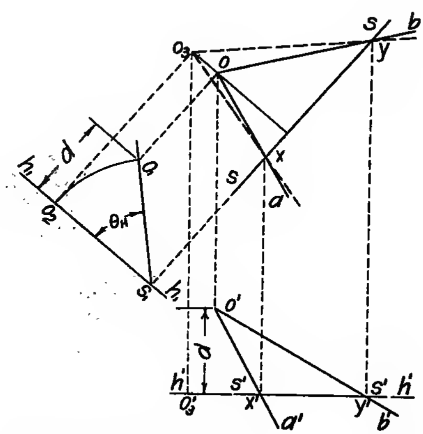


FIG. 32.—*XO_3Y true size of angle AOB.*

horizontal plane at the level of the axis. The distance d taken from the front view locates the point o_1 in the auxiliary view. The line o_1s_1 is the auxiliary view of the plane AOB and θ_H the angle through which it must be revolved to bring it into a horizontal position. When the plane is in a horizontal position, the auxiliary view of O is at o_2, its top view at o_3, and its front view

at o'_3. The points X and Y where the lines OA and OB intersect the axis do not move during the revolution. Therefore xo_3y is the true size of the angle AOB.

The problem can be solved by revolving the plane of the angle until it is parallel to V about a frontal line as an axis. Then the front view shows the true size of the angle.

42. To bisect an angle. When the angle is shown in its true size, the bisector can be drawn. The bisector can then be revolved back until the plane of the angle is in its original position and the top and front views of the bisector found. In general, the views of the bisector will not bisect the views of the angle.

The view of an angle can be larger than, equal to, or smaller than the angle itself. **The view of a right angle is a right angle when one side of the angle is parallel to the plane upon which the view is taken.**

43. By the method of the above article, the true size of any plane figure can be found if the figure is represented by two of its views. If, for example, the top and front views of a triangle or quadrilateral are given, the true size of the figure can be found by revolving its corners about any horizontal axis which lies in the plane of the figure until each corner reaches the level of the axis. Joining the corners thus located gives the true size of the figure in the top view.

44. Problems.

1. Find the true size of the angle A($-2\frac{1}{2}$, $\frac{3}{4}$, $-2\frac{1}{4}$) B(-1, $1\frac{1}{2}$, -1) C($\frac{1}{2}$, $\frac{1}{4}$, -2) by using a frontal line as an axis.

2. Draw the top and front views of the bisector of the angle B($-\frac{1}{2}$, 0, $-\frac{3}{4}$) A($-\frac{1}{2}$, $1\frac{1}{4}$ $-2\frac{1}{4}$) C($-2\frac{3}{4}$, 1, $-\frac{1}{2}$).

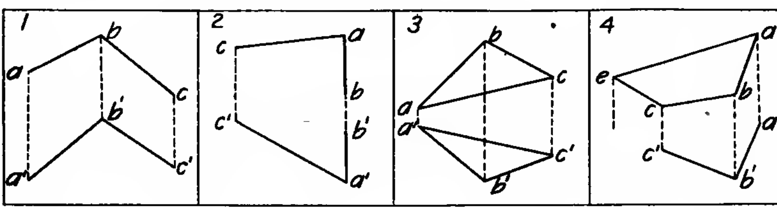
3. Draw the top and front views of the bisectors of all the angles of the triangle A($-2\frac{1}{2}$, 0, $-\frac{3}{4}$) B($-1\frac{1}{8}$, $1\frac{3}{8}$, $-2\frac{1}{8}$) C($\frac{1}{4}$, $\frac{5}{8}$, $-1\frac{3}{8}$).

4. Draw the top and front views of the quadrilateral A($\frac{1}{2}$, $1\frac{3}{4}$, $-2\frac{7}{8}$), B(0, $\frac{1}{4}$, -4) C($-1\frac{1}{2}$, 0, $-3\frac{3}{8}$) E($-2\frac{3}{4}$, $\frac{5}{8}$, ?) and find its true size and shape. (Use a 7″x 10″ rectangle for this problem.)

5. Draw the top and front views of the pyramid A($1\frac{1}{2}$, $\frac{1}{2}$, $-4\frac{1}{2}$) B($3\frac{1}{4}$, $2\frac{1}{4}$, $-4\frac{1}{2}$) C(0, $1\frac{5}{8}$, $-4\frac{1}{2}$) E($\frac{1}{2}$, 3, -2). By using an auxiliary view, find the angle between the planes AEB and ABC, the true lengths of the lines AE and BE, the slant height of the pyramid, and the true size of the face angle AEB. (Use a 7″x10″ rectangle for this problem.)

6. Find the true size of the angle between two guy wires which are anchored at A(—65′, 32.5′, —72.5′), and B(10′, —10′, —90′) and run to the top of a 50′ vertical stack with base at C(—15′, 57.5′, —82.5′). What is the length of the wires? (Use a 7″x10″ rectangle for this problem. Scale 1″ = 20′.)

7. In Art. 27, Problem 7, find the true size of one side of the roof and the angle between the hip rafters. (Use a 7″x10″ rectangle for this problem.)



45. A point and a line are represented by their top and front views. Draw the top and front views of a line which passes through the given point and makes a given angle with the given line.

Let P be the given point, 60° the given angle, and AB the given line, Fig. 33.

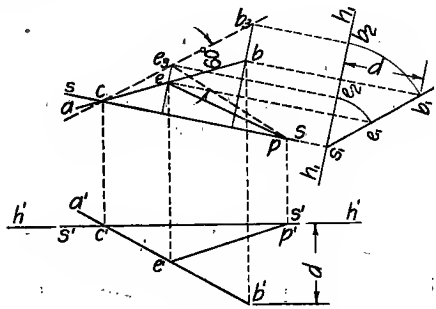


FIG. 33.—*PE 60° with AB.*

Analysis. Revolve the plane of the point and line about a horizontal axis until it is parallel to H. From the revolved position of the point, draw a line which makes the angle 60° with the revolved position of the given line. When this line is revolved back with the plane to the original position of the plane, it will be the required line.

Construction. $s's'$ is the front view and ss the top view of a horizontal line which lies on the plane of P and AB and cuts AB at the point C. When the plane is revolved about SS as an axis, B moves to B_3 while P and C remain fixed, being on the axis. b_3c is the top view of the revolved position of AB. Through p draw pe_3, making the required angle 60° with b_3c. e and e' are the top and front views of the point E after the counter revolution of the plane. Then pe is the top and $p'e'$ the front view of the required line.

46. If a plane is represented by two views of the magnitudes which determine the plane, say the top and front views of two of its lines, the views of any figure lying on this plane can be found by the method of the last article.

For example, let it be required to construct a 2″ square lying on the plane ABC, Fig. 34, having one corner of the square at A and another corner $\frac{3}{4}''$ lower than A. A horizontal line SS is first drawn, every point of which is $\frac{3}{4}''$ lower than A; $s's'$ being found first and then ss. The plane ABC is re-

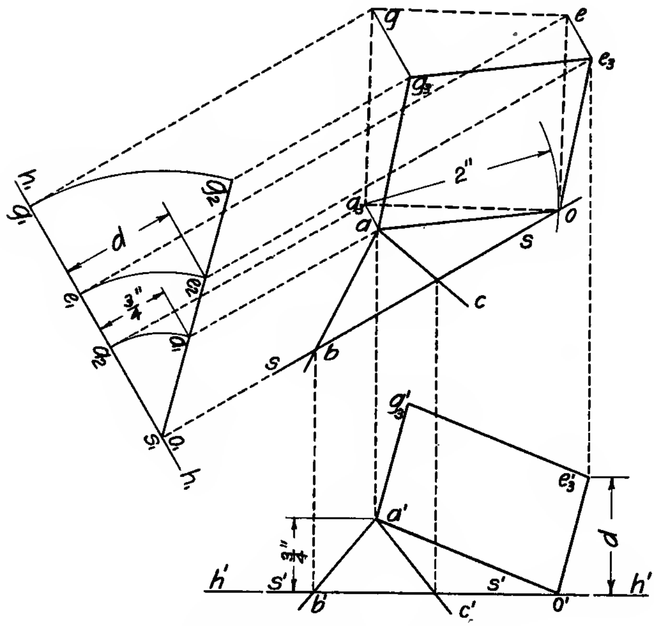


FIG. 34.—*Square on plane ABC.*

volved into a horizontal position about SS as an axis, a_3 being the top view of A after revolution. With a_3 as a center and a 2″ radius, strike an arc cutting ss at o. Construct a square with a_3o as one side. Revolve this square into the plane ABC about SS as an axis and locate its top and front views. Then AOE_3G_3 is the required square. The square could take several other positions on the plane and still satisfy the conditions of the problem.

47. Problems.

1. Draw the top and front views of a line which represents the shortest distance from the point P($-\frac{1}{2}$, 0, -1) to the line A($-\frac{1}{2}$, $1\frac{1}{4}$, $-2\frac{1}{2}$) B ($-2\frac{3}{4}$, 1, $-\frac{3}{4}$).

2. Draw the top and front views of a line which passes thru the point P(-2, $\frac{1}{2}$, $-2\frac{1}{4}$) and makes 45° with the line A(-1, $1\frac{1}{4}$, $-\frac{3}{8}$) B(-1, $\frac{1}{4}$, $-1\frac{3}{4}$).

3. Find the distance between the two parallel lines A($-\frac{3}{4}$, $1\frac{3}{8}$, $-2\frac{1}{4}$) B($\frac{3}{4}$, $\frac{5}{8}$, $-\frac{3}{4}$) and C($\frac{1}{2}$, $1\frac{3}{4}$, -2) E(2, 1, $-\frac{1}{2}$).

4. Find a point which is $2\frac{3}{8}''$ from A($-2\frac{1}{2}$, $\frac{1}{2}$, $-2\frac{1}{4}$) and lies on the line B(-1, $1\frac{1}{4}$, $-\frac{3}{4}$) C($\frac{1}{2}$, 0, $-1\frac{3}{4}$).

5. Draw the top and front views of a square with center at C($1\frac{1}{4}$, $2\frac{1}{4}$, $-2\frac{1}{2}$) and side along the line A(-1, $1\frac{3}{8}$, $-2\frac{1}{2}$) B(3, $1\frac{5}{8}$, -4). (Use a 7″ x 10″ rectangle for this problem.)

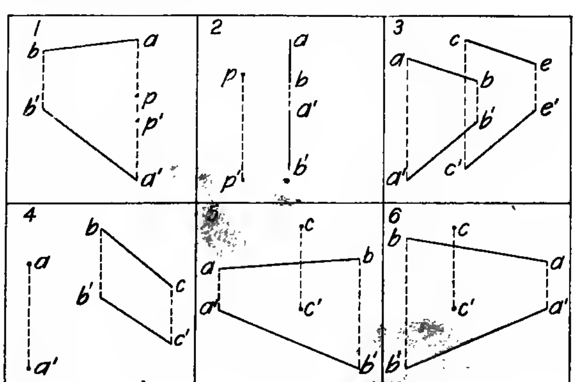
6. Draw the top and front views of an equilateral triangle with center at C($-1\frac{1}{2}$, $1\frac{3}{4}$, -3) and side along the line A($\frac{3}{4}$, 1, -3) B(-3, $1\frac{1}{2}$, $-4\frac{1}{2}$). (Use a 7″ x 10″ rectangle for this problem.)

7. A circle with center C($1\frac{1}{4}$, $2\frac{1}{4}$, $-2\frac{1}{2}$) is tangent to the line A(-1, $1\frac{3}{8}$, $-2\frac{1}{2}$) B(3, $1\frac{5}{8}$, -4). Draw its top and front views. (Use a 7″ x 10″ rectangle for this problem.

NOTE. The major axis of the top view is parallel to a horizontal line of the plane ABC, and that of the front view parallel to a frontal line of the plane.

8. A drain runs N 45° E on a falling 20% grade from the point P($-1\frac{3}{4}$, 0, -3). Draw the top and front views of the shortest connection to this drain from the point O($-\frac{3}{4}$, 2, $-2\frac{1}{2}$). What is the grade of the connection and how much pipe is required? (Scale 1″ = 20′. Use a 7″ x 10″ rectangle for this problem.)

9. What is the size of the opening which must be left in a roof whose pitch is 45° to accommodate a 16″ x 24″ chimney?



INTERSECTIONS OF LINES WITH PLANES

48. A plane is represented by two of its lines. Find the point in which a given oblique line pierce this plane.

This problem should be thoroughly mastered. It is used in finding the line of intersection of two planes, the plane sections of all ruled surfaces, the intersections of surfaces with plane faces, and indirectly in finding the intersections of such curved surfaces as cylinders and cones.

Let AB and AC, Fig. 35, be the lines which represent the plane and MN the given oblique line.

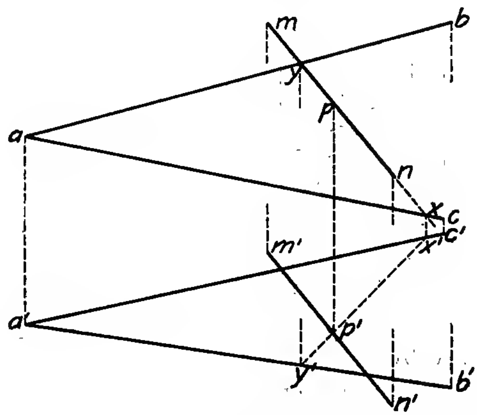


FIG. 35.—*MN pierces plane of AB and AC at P.*

Analysis. Find the points where the lines which represent the given plane pierce the plane which contains the given line and is perpendicular to H. The line joining these two points is the line of intersection of the given plane with the plane which is perpendicular to H. This line of intersection cuts the given line in the required point.

Construction. AB pierces the plane which contains MN and is perpendicular to H at Y and AC pierces it at X. $x'y'$, the front view of the line joining these two points, intersects $m'n'$ at p', the front view of the required point; p is its top view.

The point in which the given line pierces the plane ABC can be found also by means of a plane which contains MN and is perpendicular to V. The line of intersection of this plane with the plane ABC cuts MN at the piercing point.

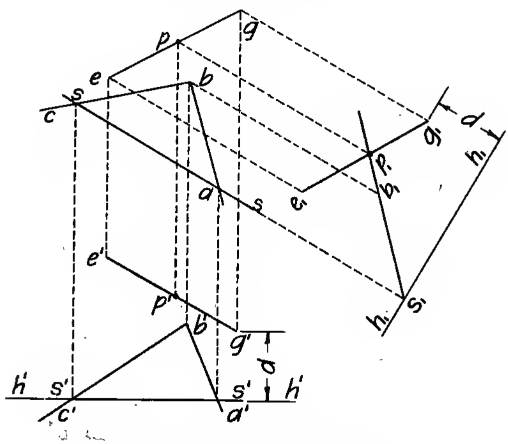


FIG. 36.—*EG pierces plane at P.*

49. Second method. Let the points A, B, and C, Fig. 36, represent the plane and EG the given oblique line.

Analysis. An auxiliary view taken by looking parallel to the plane will show the plane as one line and the oblique line as another line. Where these lines cross will be the auxiliary view of the point where the oblique line pierces the given plane. The top and front views of this piercing point can be found from the auxiliary view.

Construction. Draw the front view $s's'$ and then the top view ss of a horizontal line on the plane ABC. In the auxiliary view taken by looking in the direction of SS, h_1h_1, at right

angles to ss represents a horizontal plane at the level SS. Using this for a reference line, the lines b_1s_1 and e_1g_1 which represent respectively the plane ABC and the line EG, are determined. p_1, the intersection of b_1s_1, and e_1g_1, is the auxiliary view of the point where EG pierces the plane ABC. Since P is on the line EG, its top view is p and its front view p'.

The same piercing point could have been found by taking an auxiliary view looking in the direction of a frontal line of the plane ABC.

The first method is the better when it is required to find where one or two lines pierce a plane. The second method is preferable when it is necessary to find where a great many lines pierce one plane, as is the case with the plane section of a cone or cylinder.

50. Problems.

1. Find the point where the line E$(-1\frac{1}{4}, \frac{1}{2}, -1)$ G$(1\frac{1}{4}, 1\frac{3}{4}, -2\frac{1}{4})$ pierces the plane A$(1\frac{1}{4}, 1, -\frac{3}{4})$ B$(-1\frac{1}{4}, 1\frac{3}{4}, -1\frac{3}{4})$ C$(\frac{1}{4}, \frac{1}{4}, -2)$.

2. Find the point in which the line A$(-1\frac{1}{2}, 1\frac{1}{4}, -\frac{5}{8})$ B$(1\frac{1}{2}, \frac{1}{2}, -2\frac{1}{8})$ pierces the plane C$(-1, \frac{3}{4}, -1\frac{1}{4})$ E$(1, \frac{1}{4}, -\frac{1}{2})$ G$(-1, 1\frac{3}{4}, -2\frac{1}{4})$ K$(1, 1\frac{1}{4}, -1\frac{1}{2})$.

3. Find the point in which the line C$(-1, 1\frac{1}{8}, -2\frac{1}{4})$ E$(1\frac{1}{2}, 1\frac{1}{8}, -\frac{3}{4})$ pierces the plane A$(\frac{5}{8}, 1\frac{1}{2}, -1\frac{3}{4})$ B$(-1\frac{1}{4}, 1\frac{1}{2}, -1\frac{3}{4})$ G$(1\frac{1}{4}, \frac{1}{4}, -\frac{1}{4})$.

4. Find the point in which the line A$(-1, 1\frac{3}{4}, -\frac{3}{4})$ B$(1, \frac{1}{4}, -2\frac{1}{4})$ pierces the plane C$(-\frac{1}{2}, 1\frac{1}{2}, -2)$ E$(-2\frac{1}{2}, 1\frac{1}{2}, -2)$ G$(0, \frac{1}{4}, -\frac{1}{2})$.

5. Find the point in which the line A$(-1, 2, -2\frac{1}{4})$ B$(-1, \frac{1}{4}, -\frac{1}{2})$ pierces the plane C$(-\frac{1}{2}, 1\frac{1}{2}, -\frac{1}{2})$ E$(-2\frac{1}{2}, 1\frac{1}{2}, -2)$ G$(0, 0, -2)$.

6. Find the length of the part of the line A$(-1\frac{3}{2}, 3\frac{3}{8}, -4\frac{1}{4})$ B$(2, \frac{7}{8}, -1\frac{3}{4})$ which is included between the planes C$(\frac{1}{2}, 3, -3)$ E$(-2\frac{1}{2}, 3, -3)$ G$(-\frac{1}{4}, 1\frac{3}{8}, -4\frac{1}{8})$ and M$(-\frac{1}{4}, 2, -2)$ N$(2\frac{1}{2}, 2, -2)$ O$(1, 1, -3)$. (Use a 7″ x 10″ rectangle for this problem.)

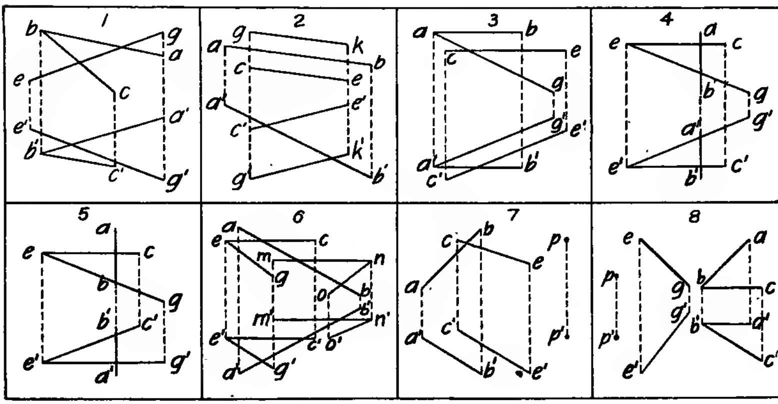
7. Draw the top and front views of a line which passes through P$(1\frac{1}{2}, 1\frac{1}{2}, -1\frac{1}{4})$ and touches the lines A$(-1\frac{1}{2}, \frac{1}{4}, -\frac{3}{4})$ B$(-\frac{1}{4}, 1\frac{3}{4}, -2)$ and C$(-\frac{3}{4}, 1\frac{5}{8}, -\frac{1}{2},)$ E$(\frac{3}{4}, \frac{7}{8}, -2)$.

8. Draw the top and front views of a line which passes through P $(-2\frac{1}{2}, \frac{1}{2}, -1)$, is parallel to the plane A$(2\frac{1}{2}, 1\frac{1}{4}, -\frac{3}{4})$ B$(\frac{1}{2}, \frac{1}{4}, -\frac{3}{4})$ C$(2\frac{3}{4}, \frac{1}{4}, -1\frac{7}{8})$, and touches the line E$(-1\frac{1}{2}, 1\frac{1}{2}, -2\frac{1}{4})$ G$(0, \frac{1}{4}, -\frac{1}{2})$.

9. A bore hole is driven at P(—250′, 100′, —100′) N45°E dipping 60°. Find the length of the bore hole to the point where it strikes the wall of a vein of ore whose strike is S(—250′, 300′, —300′) S(50′, 150′, —300′)

and whose dip is 45° to the South West. What is the difference in elevation between P and the point where the bore hole strikes the vein? (Scale 1″ = 100′. Use a 7″ x 10″ retangle for this problem).

10. Find the points in which the line A(-2, $3\frac{1}{2}$, $-3\frac{7}{8}$) B(1, $\frac{1}{2}$, $-2\frac{3}{4}$) pierces the tetrahedron whose vertex is V($\frac{1}{2}$, $2\frac{1}{2}$, $-1\frac{1}{2}$) and whose base is C($-1\frac{1}{2}$, $2\frac{1}{2}$, -4) E($-1\frac{1}{2}$, $\frac{1}{2}$, -4) G($\frac{1}{2}$, $\frac{1}{2}$, -4). (Use a 7″ x10″ rectangle for this problem).



51. Two planes are each represented by two of their lines. Find the line of intersection of the planes.

Let AB and BC, Fig, 37, be the lines of one plane and MN and OP the lines of the other plane.

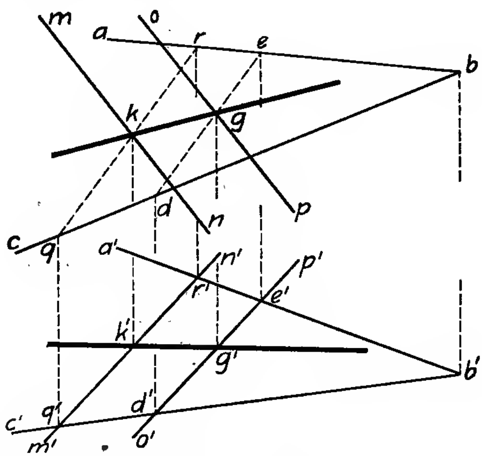


FIG. 37.—*KG line of intersection of two planes.*

Analysis. Find the points where two lines of one plane pierce the other plane. The line joining these piercing points is the required line of intersection of the planes.

Construction. MN pierces the plane of AB and BC at the point K. (Art. 48). OP pierces the plane of AB and BC at the point G. Then kg and $k'g'$ are top and front views of the required line of intersection of the planes.

Some times the lines which represent one plane do not cross the lines which represent the other plane within the limits of the drawing, or they cross in such a manner that the above method cannot be used to find the intersections of the planes. In this case points of the line of intersection can be found by drawing additional lines of one plane which do cross the lines of the other plane within the limits of the drawing. The piercing points of these additional lines determine the line of intersection of the planes.

52. Problems.

Use a 7″ x 10″ reotangle for each of the following problems:

1. Draw the top and front views of the line of intersection of the planes A(-2, 1, -3) B($\frac{1}{2}$, $3\frac{1}{2}$, $-1\frac{3}{4}$) C(-1, $\frac{1}{4}$, $-3\frac{5}{8}$) E($1\frac{1}{2}$, $2\frac{3}{4}$, $-2\frac{3}{8}$) and M (-1, $2\frac{3}{4}$, -1) N ($1\frac{1}{4}$, $1\frac{5}{8}$, $-3\frac{3}{4}$) O($-1\frac{3}{4}$, $2\frac{1}{8}$, $-1\frac{3}{4}$) P($\frac{1}{2}$, 1, $-4\frac{1}{2}$).

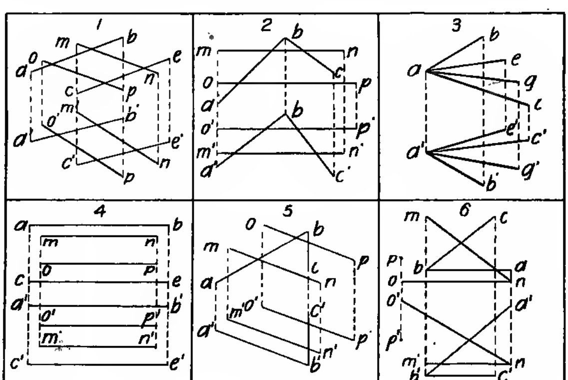
2. Draw the top and front views of the line of intersection of the planes A($-2\frac{1}{2}$, $\frac{1}{2}$, -3) B(0, 3, $-\frac{7}{8}$) C($1\frac{7}{8}$, $1\frac{3}{4}$, $-4\frac{1}{8}$) and M($-2\frac{1}{2}$, $2\frac{1}{2}$, $-2\frac{1}{2}$) N (2, $2\frac{1}{2}$, $-2\frac{1}{2}$) O($-2\frac{1}{2}$, $1\frac{1}{2}$, $-1\frac{1}{2}$) P($2\frac{1}{2}$, $1\frac{1}{2}$, $-1\frac{1}{2}$).

3. Draw the top and front views of the line of intersection of the planes B($\frac{1}{2}$, -4, -4) A(-2, $1\frac{1}{2}$, $-2\frac{3}{4}$) C(2, $\frac{1}{4}$, $-2\frac{1}{4}$) and E(1, $2\frac{1}{2}$, $-1\frac{3}{4}$) A(-2, $1\frac{1}{2}$, $-2\frac{3}{4}$) G($1\frac{1}{2}$, 1, $-3\frac{1}{2}$).

4. Draw the top and front views of the line of intersection of the planes A(-2, $2\frac{1}{2}$, $-1\frac{3}{4}$) B(2, $2\frac{1}{2}$, $-1\frac{3}{4}$) C(-2, $\frac{1}{2}$, 4) E(2, $\frac{1}{2}$, -4) and M($-1\frac{3}{4}$, 2, $-3\frac{1}{4}$) N($1\frac{3}{4}$, 2, $-3\frac{1}{4}$) O($-1\frac{3}{4}$, $1\frac{1}{4}$, $-2\frac{3}{8}$) P($1\frac{3}{4}$, $1\frac{1}{4}$, $-2\frac{3}{8}$).

5. Draw the top and front views of the lines of intersection of the planes A($-2\frac{1}{2}$, $1\frac{1}{2}$, $-2\frac{1}{2}$) B(1, $3\frac{1}{4}$, $-3\frac{3}{4}$) C(1, $1\frac{3}{4}$, $-1\frac{5}{8}$) and M(-2, $2\frac{1}{2}$, -2) N($1\frac{1}{2}$, $1\frac{1}{2}$, $-3\frac{1}{4}$) O(-1, $3\frac{1}{2}$, $-1\frac{3}{8}$) P($2\frac{1}{2}$, $2\frac{1}{2}$, $-2\frac{5}{8}$).

6. Draw the top and front views of a line which passes through the point P(-2, $1\frac{1}{2}$, $-2\frac{3}{4}$) and is parallel to both planes A(2, 1, $-1\frac{1}{4}$) B(-1, 1, $-4\frac{1}{4}$) C($1\frac{1}{2}$, $3\frac{1}{2}$, $-4\frac{1}{4}$) and M(-1, $3\frac{1}{2}$, $-3\frac{3}{4}$) N(2, $\frac{1}{2}$, $-3\frac{3}{4}$) O(-2, $\frac{1}{2}$, $-1\frac{1}{8}$).



PERPENDICULAR RELATIONS BETWEEN LINES AND PLANES

53. A line perpendicular to a plane.

A line which is perpendicular to a plane is perpendicular to every line of the plane. **Two lines which are perpendicular to each other will appear perpendicular in any view where one of the lines is shown in its true length.** Since all horizontal lines of a plane show true length in the top view, the top view of a perpendicular to a plane will, therefore, appear at right angles to the top view of all these horizontal lines. Likewise, the front view of a perpendicular to a plane will appear at right angles to the front view of all frontal lines on the plane.

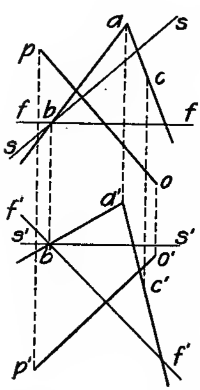


Fig. 38.—Line OP perpendicular to plane ABC.

For example, in Fig. 38, let it be required to represent a perpendicular to the plane ABC from the point P. Draw any horizontal line SS and any frontal line FF on the plane. Draw *po* perpendicular to *ss* and $p'o'$ perpendicular to $f'f'$. Then the line PO is perpendicular to the plane ABC.

In all the problems dealing with lines and planes at right angles to each other, the planes may be represented by horizontal and frontal lines unless the problem states definitely that they are to be represented otherwise. If the plane were not represented by its horizontal and frontal lines, these lines could be easily drawn, but the additional lines would complicate the drawing. Another advantage of representing a plane by its horizontal and frontal lines is that its position in space can be more easily visualized.

54. Given the top and front views of a point and of two lines which represent a plane, find the distance from the point to the plane.

Let SS and FF, Fig. 39, be the lines which represent the plane and P the given point.

Analysis. Draw a perpendicular from the given point to the given plane (Art. 53), and find where it pierces the plane (Art. 48). The true length of the perpendicular from the given point to the piercing point is the required distance.

Construction. pq, at right angles to ss, and $p'q'$, at right angles to $f'f'$, are the top and front views, respectively, of the perpendicular to the plane from the point P. In the auxiliary view taken by looking in the direction of SS, s_1f_1 represents the

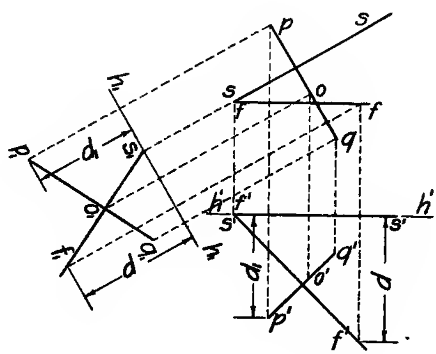


FIG. 39.—*PQ pierces plane at O.*

oblique plane and p_1 the given point. p_1q_1 at right angles to s_1f_1, is the auxiliary view of the perpendicular to the plane and o_1 is the auxiliary view of the point where it pierces the plane. o and o' are the top and front views respectively of the point where the perpendicular pierces the plane. p_1o_1 is the true distance from the point P to the given plane.

55. Conversely, to find the top and front views of a point which is a given distance from a given plane.

Let SS and FF, Fig. 40, be the lines which represent the given plane.

Analysis. There are an infinite number of points which are a given distance, say 1″, from a given plane. To represent one of

these points, select any point on the plane and erect a perpendicular to the plane at this point. On the true length of the perpendicular measure the given distance from the point where the perpendicular pierces the plane. This will locate the required point.

Construction. Take an auxiliary view of the plane from such a position that the plane appears as a straight line s_1f_1. At any point of the line s_1f_1 as x_1, erect a perpendicular to the line and measure the given distance along the perpendicular from the point

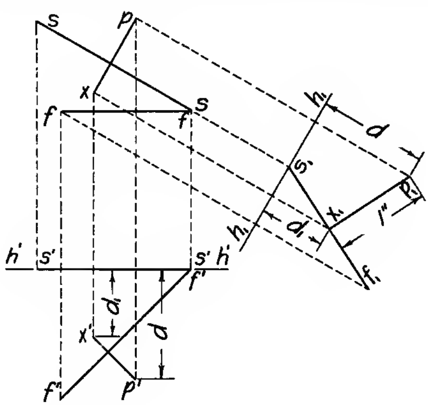


FIG. 40.—*P is 1" from plane.*

x_1, thus locating P_1, the auxiliary view of the required point. The top view of the point X is at any point along the line through x_1 parallel to ss as x. A perpendicular to ss from x, and a parallel to ss from p_1 intersect at p, the top view of the required point 1" from the plane. p' is its front view. $p'x'$ should check perpendicular to $p'f'$.

This construction can be used to locate the vertex of a right cone or pyramid or the upper corners of a rectangular object when the base of the object is in an oblique plane.

56. Given the top and front views of a point and of an oblique line. Represent a plane which contains the point and is perpendicular to the line.

Let P, Fig. 41, be the given point and AB the given line.

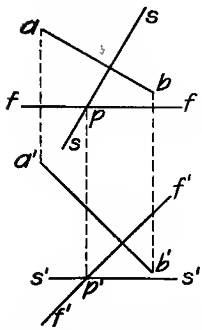


FIG. 41,—*Plane perpendicular to AB.*

Analysis. Since the plane is to be perpendicular to the line, a horizontal line of the plane will have its top view perpendicular to the top view of the line. Likewise, a frontal line of the plane will have its front view perpendicular to the front view of the line. Therefore to represent the plane, draw the top and front views of a horizontal line and of a frontal line which pass through the given point and are perpendicular to the given line. These two lines will represent the required plane.

Construction. Through p, draw ss perpendicular to ab and through p' draw $s's'$ parallel to G. L. Through p', draw $f'f'$ perpendicular to $a'b'$ and through p draw ff parallel to G. L. Then the plane of SS and FF is perpendicular to the line AB.

Although the plane of SS and FF is perpendicular to the line AB, the line AB does not intersect either of the lines SS or FF.

57. Problems.

1. Find the distance from the point P($\frac{1}{4}$, $1\frac{7}{8}$, $-\frac{3}{4}$) to the plane A($-2\frac{1}{4}$, $\frac{1}{2}$, $-\frac{3}{4}$) B($-\frac{1}{4}$, $1\frac{1}{2}$, $-2\frac{1}{4}$) C($1\frac{1}{4}$, 0, $-1\frac{3}{4}$.)

2. Find the distance from the point O($-\frac{1}{2}$, $1\frac{3}{4}$, $-\frac{1}{2}$) to the plane A(-2, $1\frac{3}{4}$, -2) B($\frac{1}{2}$, $1\frac{1}{4}$, -2) C(-2, $\frac{1}{4}$, -1) E($\frac{1}{2}$, $\frac{1}{4}$, -1.)

3. O($-1\frac{5}{8}$, $\frac{3}{8}$, -2) is one corner of a cube which has its base in the plane C($-2\frac{3}{4}$, $1\frac{1}{2}$, $-\frac{1}{2}$,) A($-\frac{1}{2}$, $1\frac{3}{4}$, $-2\frac{1}{4}$) B($-\frac{1}{2}$, $\frac{1}{2}$, $-\frac{3}{4}$). Find the size of the cube.

4. B is one corner of a $1\frac{1}{2}''$ cube which has its base in the plane A($-2\frac{1}{2}$, 1, $-2\frac{1}{4}$) B(-1, $1\frac{3}{4}$, $-\frac{3}{4}$) C($\frac{1}{2}$, $\frac{1}{2}$, $-1\frac{2}{4}$). Locate an adjacent corner which does not lie in the plane of the base.

5. Represent a plane which is parallel to and is $\frac{3}{4}''$ from the plane A($-2\frac{1}{2}$, $\frac{1}{4}$, $-\frac{7}{8}$) B($-1\frac{1}{8}$, $1\frac{5}{8}$, $-2\frac{1}{8}$) C($\frac{1}{4}$, $\frac{7}{8}$, $-1\frac{1}{2}$.)

6. Locate a point which is $1''$ from the plane A($-\frac{3}{4}$, $1\frac{3}{8}$, $-2\frac{1}{4}$) B($\frac{3}{4}$, $\frac{5}{8}$, $-\frac{3}{4}$) C($\frac{1}{2}$, $1\frac{3}{4}$, -2) E(2, 1, $-\frac{1}{2}$.)

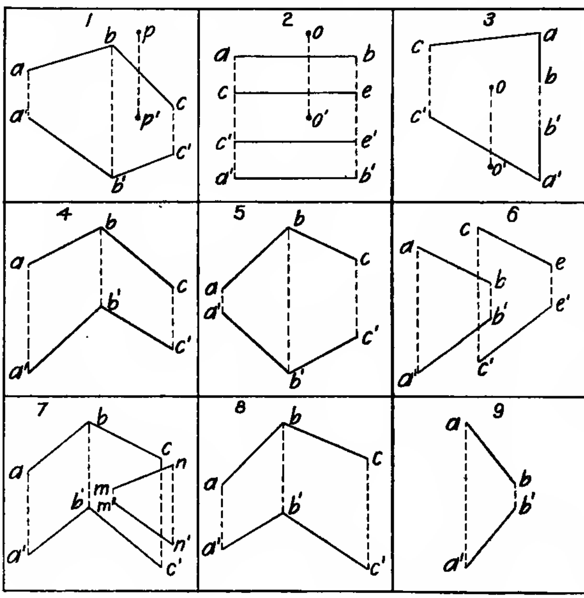
7. Represent a plane which contains the line M($\frac{1}{4}$, 1, $-\frac{3}{4}$) N($1\frac{3}{4}$, 2, $-2\frac{1}{4}$) and is perpendicular to the plane A($-1\frac{3}{4}$, 2, -3) B($-\frac{1}{4}$, $3\frac{1}{2}$, -2) C($1\frac{1}{4}$, $2\frac{1}{2}$, -4). (Use a $7''$ x $10''$ rectangle for this problem.)

8. Find the locus of all points which are equidistant from the points A($-\frac{1}{2}$, $1\frac{1}{2}$, -1) B(1, 3, -2) and C($2\frac{1}{2}$, 2, -4). (Use a 7″ x 10″ rectangle for this problem.)

9. A(-1, $1\frac{3}{4}$, $-2\frac{1}{4}$) B(0, $\frac{1}{4}$, $-\frac{3}{4}$) is one edge of a cube. Represent by two lines the plane of the base of the cube.

10. Find the center of the circle in which P ($-\frac{1}{2}$, $\frac{1}{2}$, -1) moves when it rotates about A(0, $1\frac{3}{4}$, $-2\frac{1}{4}$) B(1, $\frac{1}{4}$, $-\frac{3}{4}$) as an axis.

11. A(-2, $2\frac{1}{4}$, -3) B(-1, $\frac{1}{2}$, $-1\frac{1}{4}$) is the base of an isosceles triangle which has its vertex in the line C($\frac{1}{2}$, $\frac{1}{2}$, -3) E(2, $3\frac{3}{4}$, $-2\frac{1}{2}$). Represent the triangle in the top and front views. (Use a 7″ x 10″ rectangle for this problem.)



58. Given the top and front views of two lines of a plane and the top and front views of a third line which does not lie on this plane. Find the projection of the third line on the given plane.

Let SS, and FF, Fig. 42, represent the given plane and AB any line which does not lie on this plane.

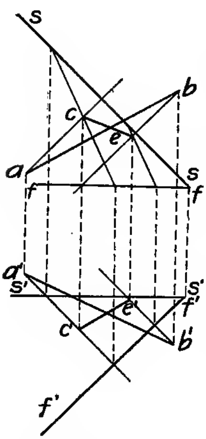


FIG. 42.—*CE projection of AB on plane.*

Analysis. From any two points of the given line, erect perpendiculars to the given plane. A line joining the points in which these perpendiculars pierce the plane will be the required projection. The point in which the given line pierces the given plane is also a point on the projection of the line on the plane.

Construction. The top view of the perpendicular to the plane SSFF from the point B is the line through *b* perpendicular to *ss*. The front view of this perpendicular is the line through *b′* perpendicular to *f′f′*. This perpendicular pierces the given plane at E. The perpendicular to the plane from A pierces the given plane at the point C. Then *ce* is the top and *c′e′* the front view of the projection of AB on the plane SSFF.

59. Problems.

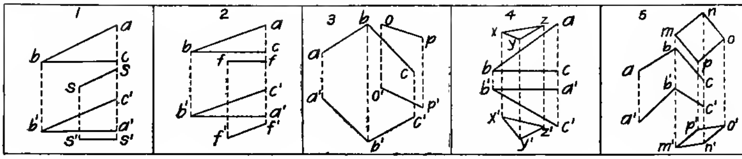
1. Find the projection of the line S($\frac{1}{2}$, 0, —2$\frac{1}{4}$) S(1$\frac{1}{2}$, $\frac{1}{2}$, —2$\frac{1}{4}$) on the plane A(1$\frac{1}{2}$, 2$\frac{1}{8}$, —2) B(—1$\frac{1}{4}$, $\frac{3}{4}$, —2) C(1$\frac{1}{2}$, $\frac{3}{4}$, —$\frac{3}{8}$).

2. Find the projection of the line F($\frac{1}{2}$, $\frac{1}{4}$, —2$\frac{1}{4}$) F(1$\frac{1}{2}$, $\frac{1}{4}$, —1$\frac{3}{4}$) on the plane A(1$\frac{1}{2}$, 2$\frac{1}{8}$, —1$\frac{1}{2}$) B(—1$\frac{1}{4}$, $\frac{1}{2}$, —1$\frac{1}{2}$) C(1$\frac{1}{2}$, $\frac{1}{2}$, —$\frac{1}{8}$).

3. Find the projection of the line O($\frac{1}{8}$, 3$\frac{1}{4}$, —1) P(1$\frac{1}{2}$, 2$\frac{3}{8}$, —2$\frac{3}{8}$) on the plane A(—1$\frac{3}{4}$, 1$\frac{1}{2}$, —1$\frac{1}{2}$) B(—$\frac{1}{4}$, 3, —3$\frac{1}{2}$) C(1$\frac{1}{4}$, $\frac{1}{2}$, —2$\frac{1}{2}$). (Use a 7″x10″ rectangle for this problem).

4 Find the projection of the triangle X(—1, 2$\frac{3}{8}$, —3) Y(—$\frac{1}{8}$, 2$\frac{1}{8}$,—4$\frac{1}{8}$) Z($\frac{3}{4}$, 2$\frac{5}{8}$, —3$\frac{1}{8}$) on the plane A(1$\frac{1}{2}$, 3, —1) B(--1$\frac{1}{2}$, $\frac{1}{2}$, —1) C(1$\frac{1}{2}$, $\frac{1}{2}$, —3). (Use a 7″x10″ rectangle for this problem).

5. Find the projection of the parallelogram M$(0, 3, -3\frac{3}{4})$ N$(1, 3\frac{3}{4}, -3\frac{1}{2})$ O$(1\frac{3}{4}, 2\frac{3}{4}, -3)$ P$(\frac{3}{4}, 2, -3\frac{1}{4})$ on the plane A$(-1\frac{3}{4}, 1\frac{1}{4}, -3)$ B$(0, 2\frac{1}{2}, -1)$ C$(1\frac{1}{2}, \frac{1}{2}, -2)$. (Use a 7″ x 10″ rectangle for this problem).



60. Given the top and front views of two lines of a plane and the top and front views of a third line which does not lie on this plane, find the angle which the third line makes with the given plane.

Let SS and FF, Fig. 43, represent the given plane and AB any line which does not lie on this plane.

Analysis. The angle which a line makes with a given plane is understood to be the angle which the line makes with its projection on that plane. If a perpendicular be dropped to the plane from any point in the given line, the angle between this perpendicular and the given line is the complement of the angle which the line makes with the plane. Therefore, find the angle between the given line and a perpendicular to the plane from any point of the line construct its complement. This complement is the required angle.

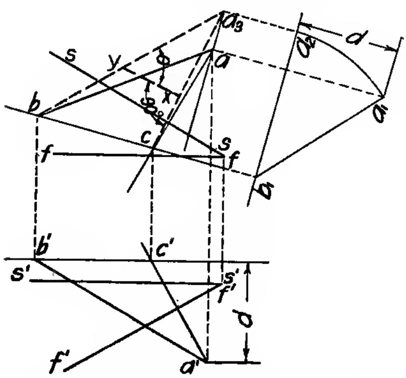


FIG. 43.—θ *angle AB makes with plane.*

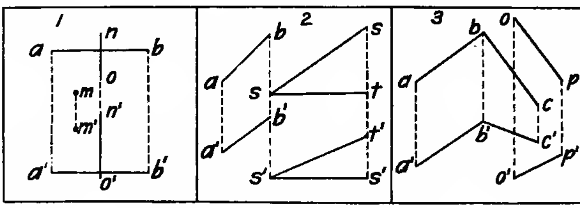
Construction. Draw ac and $a'c'$ perpendicular respectively to a horizontal line and a frontal line of of the plane SSFF. BC is a horizontal line on the plane BAC. When A is rotated about BC it reaches the level of the axis at a_3. ba_3c is the true size of the angle between AB and the perpendicular to the plane from A. At any point x of the line ca_3, draw xy perpendicular to ca_3. Then xya_3 is the true size of the angle which the line AB makes with the plane SSFF.

61. Problems.

1. Find the angle which the line A(-2, $1\frac{1}{8}$, $-2\frac{1}{8}$) B(0, $1\frac{1}{8}$, $-2\frac{1}{8}$) makes with the plane M($-1\frac{1}{2}$, $\frac{1}{4}$. $-1\frac{1}{8}$) N(-1, $1\frac{3}{4}$, $-\frac{5}{8}$) O(-1, $\frac{1}{2}$, $-2\frac{1}{4}$).

2. Find the angle which the line A($-1\frac{1}{2}$, $\frac{1}{2}$, $-1\frac{1}{4}$) B($-\frac{1}{2}$, $1\frac{1}{2}$, $-\frac{3}{4}$ makes with the plane S($2\frac{1}{2}$, $1\frac{3}{4}$, -2) S($\frac{1}{2}$, $\frac{1}{4}$, -2) T($2\frac{1}{2}$, $\frac{1}{4}$, -1).

3. Find the angle which the line O($-\frac{1}{2}$, $3\frac{1}{4}$, $-4\frac{1}{4}$,) P(1, $1\frac{1}{4}$, -3) makes with the plane A($-3\frac{1}{4}$, $1\frac{1}{4}$, -3) B($-1\frac{1}{2}$, $2\frac{1}{2}$, -1) C(0, $\frac{1}{2}$, -2). (Use a 7" x 10" rectangle for this problem).



62. Two planes are each represented by the top and front views of two of their lines, find the angle between the planes.

Let ABC and MNO, Fig. 44, be the given planes.

Analysis. From any point in space drop a perpendicular to each plane. The angle between these perpendiculars is the same size as the angle between the planes.

Construction. From any point P, draw PE, perpendicular to the plane ABC and PG perpendicular to the plane MNO. SS is a horizontal line in the plane of the perpendiculars. By rotating the perpendiculars about SS as an axis, the true size of the angle between them is found to be ep_3g. Therefore ep_3g is the true size of the angle between the planes ABC and MNO.

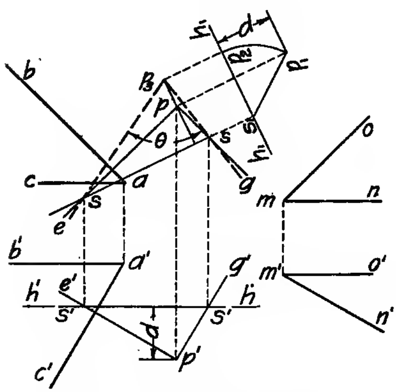


FIG. 44.—θ angle between planes.

If an auxiliary view of the two planes be taken by looking in the direction of their line of intersection, each plane appears as a line and the angle between the lines is the true size of the angle between the planes. If one of the planes is parallel to H or V, the auxiliary view method (Arts. 34 and 37) is the best to use. If the line of intersection of the planes is oblique to both H and V. the auxiliary view taken by looking in the direction of this line cannot be constructed directly from the top and front views. In this case an auxiliary view can be taken by looking parallel to H and at right angles to the line of intersection of the planes. Then from this view a second auxiliary view can be constructed by looking in the direction of the line of intersection. The second auxiliary view shows the true size of the angle between the planes. The construction for this method is more

difficult than for the method using the perpendiculars as explained above.

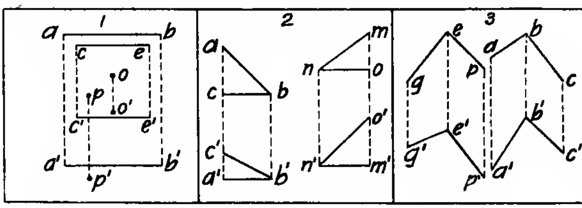
63. Problems.

1. Find the angle between the planes A($-1\frac{1}{2}$, $1\frac{3}{4}$, -2) B($\frac{1}{2}$, $1\frac{3}{4}$, -2) O($-\frac{1}{2}$, $\frac{3}{4}$, $-\frac{1}{2}$) and C($-1\frac{3}{4}$, $1\frac{1}{2}$, $-\frac{3}{4}$) E($\frac{1}{4}$, $1\frac{1}{2}$, $-\frac{3}{4}$) P(-1, $\frac{1}{4}$, $-2\frac{1}{4}$).

2. Find the angle between the planes A(-3, $1\frac{3}{4}$, $-2\frac{1}{4}$) B($-1\frac{1}{2}$, $\frac{1}{4}$, $-2\frac{1}{4}$) C(-3, $\frac{1}{4}$, $-1\frac{1}{2}$) and M(3, $1\frac{3}{4}$, -2) N($1\frac{1}{2}$, $\frac{3}{4}$, -2) O(3, $\frac{3}{4}$, $-\frac{5}{8}$).

3. Find the angle between the planes G($-3\frac{1}{4}$, 2, $-3\frac{1}{2}$) E($-1\frac{3}{4}$, $3\frac{1}{2}$, $-2\frac{1}{2}$) P($-\frac{1}{4}$, $2\frac{1}{2}$, $-4\frac{1}{2}$) and A(0, $2\frac{3}{4}$, $-4\frac{1}{4}$) B($1\frac{3}{4}$, 4, $-2\frac{1}{4}$) C($3\frac{1}{4}$, 2, $-3\frac{1}{4}$). (Use a 7″x10″ rectangle for this problem).

4. A hip roof with peak at O($1\frac{1}{2}$, 2, $-2\frac{3}{4}$), and plate corners at A($-\frac{1}{2}$, $\frac{1}{2}$, -4) B($-\frac{1}{2}$, $3\frac{1}{2}$, -4) C($2\frac{1}{2}$, $3\frac{1}{2}$, -4) and E($2\frac{1}{2}$, $\frac{1}{2}$, -4) is covered with tile. What should be the angle between the faces of the tile directly over the hip rafter? (Use a 7″x10″ rectangle for this problem).



64. Two oblique lines which do not lie in the same plane are given by their top and front views, find the top and front views and the true length of their common perpendicular.

General Analysis. An auxiliary view taken from such a position that one of the given lines appears as a point, shows the common perpendicular in its true length. If one of the given lines is parallel to either H or V, but one auxiliary view is necessary to show this line as a point, Case 1. If both of the given lines are oblique to H and V, two successive auxiliary views are necessary to show one of the lines as a point, Case 2.

Case 1. When one of the lines is parallel to either H or V.

Let AB and SS, Fig. 45, be the given lines.

Analysis. Take an auxiliary view of the lines by looking in the direction of the line which is parallel to H. In the auxiliary

view, this line appears as a point and the other one as a line. A perpendicular from the point to this line is the auxiliary view of the common perpendicular to the two given lines, and is the true length of the perpendicular. From the auxiliary view of the perpendicular, its top and front views can be found.

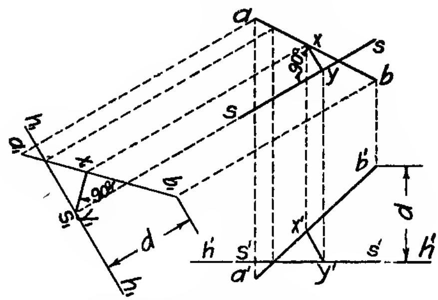


FIG. 45.—*XY common perpendicular to AB and SS.*

Construction. s_1 and a_1b_1 are the auxiliary views of the given lines taken by looking in the direction of SS; h_1h_1 represents the horizontal plane at the level of SS. x_1y_1, perpendicular to a_1b_1, is the auxiliary view of the common perpendicular and is its true length. x is the top view of the point X. xy, perpendicular to ss, is the top and $x'y'$ the front view of the required perpendicular to the lines AB and SS.

When one of the given lines is parallel to V instead of H as above, the auxiliary view should be taken by looking in the direction of this line.

65. Case 2. When both of the lines are oblique to H and V. Let AB and CE, Fig. 46, be the given lines.

Analysis. Take a view of both lines by looking parallel to H and perpendicular to one of the lines. Using this auxiliary view in connection with the top view, a second auxiliary view can be drawn. The second auxiliary view is taken by looking in the direction of the line which is shown in its true length in the first auxiliary view. In the second auxiliary view, one line appears

as a point and the other as a line. From the point draw a perpendicular to the line. This is the second auxiliary view of the required perpendicular and shows its true length. From this view the first auxiliary, the top, and the front views of the perpendicular can be drawn.

Construction. The first auxiliary view is taken by looking parallel to H and perpendicular to the line AB. a_1b_1 and c_1e_1 are the auxiliary views of the given lines. Consider this auxiliary view as a new front view and disregard for the time being the original front view. The second auxiliary view is taken from the first by looking in the direction of AB. In this view AB appears as the point a_2b_2 and CE appears as the line c_2e_2. From the point a_2b_2, draw the line x_2y_2 perpendicular to the line c_2e_2. x_2y_2 is the second auxiliary view of the common perpendicular to the two lines and is its true length. From the point y_2, draw a line parallel to a_1b_1 intersecting c_1e_1 at y_1. Draw y_1x_1 perpendicular to a_1b_1. x_1y_1 is the first auxiliary view of the common perpendicular. From this view the top view xy and then the front view $x'y'$ of the common perpendicular can be drawn.

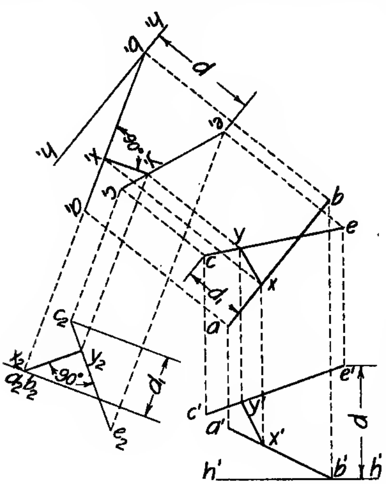


FIG. 46.—*XY common perpendicular to AB and CE.*

The distances, as d, used in locating points in the first auxiliary view are measured in the direction of the line joining the top and front views of the point. In a similar manner, the distances, as d_1, used in locating points in the second auxiliary view are measured in the direction of the line joining the top and first auxiliary views of a point. In general, distances used in locating points in any view are measured in the direction of the line join-

ing the views of a point in the two next preceding views. This process can be continued indefinitely.

66. Problems:

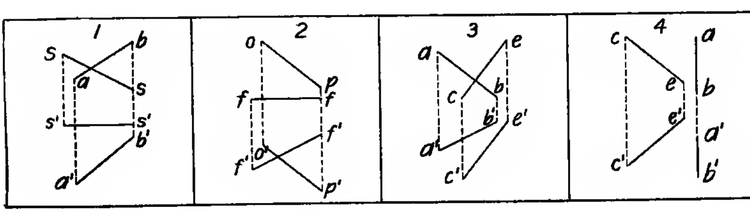
1. Draw the top and front views of the common perpendicular to the lines S($-1\frac{1}{2}$, $1\frac{1}{4}$, -1) S(0, $\frac{1}{2}$, -1) and A($-1\frac{1}{4}$, $\frac{3}{4}$, $-2\frac{1}{4}$) B(0, $1\frac{1}{2}$, $-1\frac{1}{4}$).

2. Draw the top and front views of the common perpendicular to the lines F($-1\frac{1}{2}$, $\frac{1}{2}$, -2) F(0, $\frac{1}{2}$, $-1\frac{1}{4}$) and O($-1\frac{1}{4}$, $1\frac{3}{4}$, $-1\frac{1}{2}$) P(0, $\frac{3}{4}$, $-2\frac{1}{4}$).

3. Find the top and front views of the common perpendicular to the lines A($-\frac{1}{4}$, $1\frac{3}{4}$, $-1\frac{5}{8}$,) B(1, $\frac{3}{4}$, -1) and C($\frac{1}{4}$, $\frac{3}{4}$, $-2\frac{1}{4}$) E($1\frac{1}{4}$, 2, -1).

4. Find the top and front views of the common perpendicular to the lines A($\frac{1}{4}$, $1\frac{1}{2}$, $-1\frac{1}{2}$) B($\frac{1}{4}$, $\frac{1}{4}$, $-2\frac{1}{4}$) and C($-\frac{3}{4}$, $1\frac{1}{2}$, -2) E(0, $\frac{1}{2}$, $-1\frac{1}{4}$).

5. Find the length and location of the shortest connection which can be made between two pipes with center lines A(—17′, 11′, —29′) B(—6′, 0′, —34′) and C(—14′, 8′, —20′) E(0′, 10′, —36′). (Use center lines only. Scale $\frac{1}{8}$″=1″ —0″. Use a 7″x10″ rectangle for this problem).



SPECIAL PROBLEMS

67. To show the top and front views of a line AB which makes 30° with H and 45° with V.

If the line is first taken parallel with V and 30° with H, it will have ab_1 for a top and $a'b'_1$ for a front view. $a'b'_1$ is the true length of the part of the line under consideration. If a line of the same length is placed parallel with H and 45° with V, it would have ab_2 for a top and $a'b'_2$ for a front view. If the line AB_1 be rotated about an axis through A perpendicular to H, its top view would remain a constant length. b_1 would move in the arc of a circle b_1b, and b'_1 would move along the straight line b'_1b' parallel to G. L. The angle which the line AB_1 makes with H would remain 30°, but the angle which it makes with V would increase as B_1 moves away from V. If the line AB_2 be rotated about an axis through A perpendicular to V, its front view would remain a constant length. b'_2 would move in the arc of a circle b'_2b' and b_2 would move along the straight line b_2b parallel to G. L. The angle which the line AB_2 makes with V would remain 45°, but the angle which it makes with H would increase B_2 moves away from H. The circles in which B_1 and B_2 move intersect at the point B, the line bb' being perpendicular to G. L. These circles would not have intersected if the lines AB_1 and AB_2 had been of different lengths. Then ab is the top and $a'b'$ the front view of a line which makes 30° with H and 45° with V.

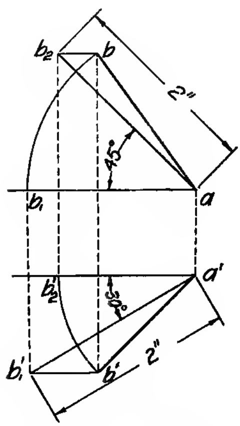


FIG. 47.—*AB 30° with H and 45° with V.*

The stationary point A of the line can be selected at random. There are four lines which contain this point A and make the given angles with H and V.

The sum of the angles which a line AB makes with H and V can vary from 0° to 90°. When the sum is 0° the line is par-

allel to G. L.; when it is 90° the line is perpendicular to G. L., that is, a line of profile.

68. To represent a plane which makes 45° with H and 60° with V.

The angles which a plane makes with H and V are the complements respectively of the angles which a line perpendicular to the plane makes with H and V. Therefore to represent a plane which makes 45° with H and 60° with V, first represent a line AB which makes 45° with H and 30° with V (Art. 67). Then represent a plane which is perpendicular to the line AB (Art. 56). This is the required plane.

The required plane can pass through any point in space and make the given angles with H and V.

The sum of the angles which the plane makes with H and V can vary from 90° to 180°.

GENERAL PROBLEMS IN POINT, LINE, AND PLANE

In the following list of problems, those marked (*a*) can be conveniently solved in a 7″x10″ rectangle, those marked (*b*) in a 10″x14″ rectangle. In each case the base line should be drawn through the center of the rectangle and parallel to its short side.

1. (*a*) The top view of a parallelogram ABCE is a 2″ square. Draw the front view of the parallelogram when its center is 1″ above, and the corner B $\frac{1}{2}$″ below the corner A. Find the true size of the parallelogram.

2. (*a*) Draw the top and front views of a line which passes through the point P(0, 2, —2), is parallel to the plane A(—$\frac{1}{2}$, $\frac{1}{2}$, —2) B(2, 3, —2) C(2, $\frac{1}{2}$, —3$\frac{1}{2}$), and has its top and front views parallel to each other.

3. (*a*) The top views of two points are 1″ apart and their front views are 2″ apart. What is the greatest and what is the least distance apart that the points can be placed?

4. (*a*) Draw the top and front views of the locus of a point which is 3″ from A(—2$\frac{1}{2}$, $\frac{1}{2}$, —2) and 2″ from B($\frac{1}{2}$, 1$\frac{1}{2}$, —4).

5. (*a*) Find a point P on the line M(1$\frac{1}{4}$, 1$\frac{1}{8}$, —$\frac{3}{4}$) N(—$\frac{1}{2}$, 2$\frac{1}{4}$, —2$\frac{1}{2}$) which is equidistant from the planes A(—1$\frac{1}{2}$, 1, —1) B(1$\frac{1}{2}$, 3, —1) C(1$\frac{1}{2}$, 1, —1) and A(—1$\frac{1}{2}$, 1, —1) B(1$\frac{1}{2}$, 3, —1) E($\frac{1}{2}$, 1, —3).

6. (*a*) Find the top and front views of a point O which is equidistant from A(—1$\frac{1}{2}$, 1, —1) B(1$\frac{1}{2}$, 1, —3) C(1$\frac{1}{2}$, 3, —1) and E(—$\frac{1}{2}$, 3, —2).

7. (*a*) Represent a plane which is equidistant from a point A(1$\frac{1}{2}$, 2, —1) and a line B(—2, 3, —1) C($\frac{1}{2}$, 0, —3) and is parallel to the line E(2, 3, —1) F(—1, 0, —3).

8. (*a*) Represent a plane which is equidistant from the points A(1$\frac{1}{2}$, 2, —1), B(—2, 3, —1), and C($\frac{1}{2}$, 0, —3) and is parallel to the line E(2, 3, —1) F(—1, 0, —3). (Three solutions).

9. (*a*) Through the points A(1, 1, —2), B(—1, 2, —1), and C(0, 3, —3), draw three planes which are parallel and equidistant.

10. (*a*) Through the points A(1, 1, —2), B(—1, 2, —1), and C(0, 3, —3), draw three planes which are parallel and equidistant, and such that one of the planes passes through P(—2, 1, —2).

11. (*a*) Through the points A(0, 2, 0), B(2, 1, —3), C(—1, 0, —3), and E(—1$\frac{1}{2}$, 2$\frac{1}{2}$, —2), draw four planes which are parallel and equidistant.

12. (*a*) Draw, respectively, through the points A(1, 1, —2) and B(—1, 2, —1) and the line C(0, 3, —3) E($\frac{1}{2}$, 0, —1), three planes which are parallel and equidistant.

13. (*a*) Draw the top and front views of a line which touches the three lines: A(—1, 1, —1) B (1, 3, —3), C(—2, 2, 0) E(0, 1, —3), and F(0, 3, 0) G(2, 1, —2).

14. (*a*) Draw the top and front views of a line which passes through

$P(2\frac{1}{2}, 3, 0)$ and touches the lines $C(-2, 2, 0)$ $E(0, 1, -3)$ and $F(0, 3, 0)$ $G(2, 1, -2)$.

15. (*a*) Through the lines $A(-1, 1, -1)$ $B(1, 3, -3)$, $C(-2, 2, 0)$ $E(0, 1, -3)$, and $F(0, 3, 0)$ $G(2, 1, -2)$, draw three planes which have a common line of intersection.

16. (*a*) Draw the top and front views of a plane which contains $A(-\frac{1}{2}, 0, -1)$ $B(-2, 1\frac{1}{2}, -2)$ and passes 1″ from $P(0, 2, -2)$.

17. (*a*) $A(-1, 0, -1)$ $B(1, 1, ?)$ $C(?, ?, -3)$ is a 3″ equilateral triangle. Draw its top and front views.

18. (*a*) Draw the top and front views of a triangle $A(-\frac{1}{2}, 1, -2)$ $B(1\frac{1}{2}, 2, -4)$ $C(?, ?, ?)$. AC=3″, BC=2″, and the point C lies in the plane $E(0, 0, -2)$ $F(3, 0, -4)$ $G(3, 2, -2)$.

19. (*a*) Draw the top and front views of the line MN which intersects the lines $A(1, 2, -3)$ $B(3, 1, -3\frac{1}{2})$ and $A(1, 2, -3)$ $C(0, 1, -4)$ and is parallel to and 1″ from the plane $E(0, 0, -2)$ $F(3, 0, -4)$ $G(3, 2, -2)$.

20. (*a*) Draw the top and front views of the triangle $A(1\frac{1}{2}, 0, -2\frac{1}{2})$ $B(-1, ?, -2)$ $C(?, ?, 0)$, when AB=$2\frac{3}{4}$″, BC=3″, and AC=$3\frac{1}{2}$″.

21. (*a*) Locate the center and find the diameter of a sphere which is tangent to planes through the origin parallel to H and V, and has its center on the line $A(-2, 2, 0)$ $B(2, 0, -3)$.

22. (*a*) At a point of outcrop $O(0, 1, -1)$ the strike of a body of ore is north 45° east. The "dip" (the inclination of the face of the ore body to the horizontal) is 60° S. E. A tunnel is driven at $P(-1, 2, -3)$ south 60° east on a rising 10% grade. How far must the tunnel be driven to reach the ore body?

NOTE. The "strike" means a horizontal line in the plane of the face of the ore body.

23. (*a*) Two ore veins have strike lines due N. E. and both veins dip 30° S. E. The strike lines are 200′ apart. What is the shortest distance between the veins?

24. (*a*) From the point of outcrop $A(0, 3, 0)$, run south 30° west 100′ to a drillhole B. The drillhole C is 100′ from both A and B. The drill strikes ore at B at a depth of 25′ and at C at a depth of 50′. Determine the strike and dip of the vein.

25. (*a*) The strike of a vein passes through the origin and bears north 75° east. The vein dips 60° toward the north and west. A drillhole which passes through the origin, bears north 30° east and dips 75°. If the core from the drillhole shows 6′ of vein matter, what is the true thickness of the vein? Scale $\frac{1}{2}$″=1′—0″.

26. (*a*) The strike of a vein passes through the origin and bears north 45° east. A drillhole through the origin bears north 15° east and dips 75°.

If the core from the drillhole shows 6′—0″ of vein matter and the vein is known to be 3′—0″ thick, what must be the dip of the vein? Scale $\frac{1}{2}''=1'—0''$.

27. (*b*) Given a plane M(—2, 2, 0) N(1,5, 0) O(1, 2, —3) and a line A($2\frac{3}{4}$, $\frac{1}{4}$, 0) B($\frac{3}{4}$, $1\frac{3}{8}$, 0). A square which has AB for one side lies in a horizontal plane. Revolve this square about MN as an axis until it comes into the plane MNO and then draw the top and front views of a cube which has this square for a base.

28. (*b*) Given a plane MNO as in problem 27. A $2\frac{1}{2}''$ cube rests on this plane in such a position that two edges AB and EF of the base make angles of 15° with the horizontal line MN. The corner B is at the point (—1, ?, —1). Draw the top and front views of the cube.

29. (*b*) Keeping the edge BA as in 28, draw the top and front views of the same cube with one corner in H. The entire base of the cube does not rest on the plane MNO.

30. (*b*) Keeping the edge BA as in 28, draw the top and front views of the cube when one of its corners is in a plane parallel to and $1\frac{1}{4}''$ from V.

31. (*b*) Keeping the corner B as in 28, draw the top and front views of the cube with one corner in H and another in V.

32. (*b*) One side of an equilateral triangle lies along the line A(—$\frac{1}{2}$, $3\frac{3}{4}$, —$2\frac{3}{8}$) B($3\frac{3}{4}$, $1\frac{1}{2}$, —$5\frac{3}{4}$) and the vertex opposite this side is at the point C(3, 5, —3). Draw the top and front views of a regular pyramid having this triangle for a base and an altitude of $4\frac{1}{2}''$.

33. (*b*) A regular triangular pyramid whose altitude is 5″ stands on a plane which contains M(—2, 4, —$4\frac{1}{4}$) N($2\frac{3}{4}$, $1\frac{1}{4}$ —$4\frac{1}{4}$) and slopes downward to the back making 45° with H. Draw the top and front views of the pyramid when one corner of the base lies in the line A(—$2\frac{1}{4}$, 3, —$2\frac{1}{8}$) B(—$\frac{3}{4}$, $1\frac{1}{2}$, —$5\frac{1}{2}$) and one side of the base lies along the line of intersection of the plane of the base and a plane perpendicular to AB at a point 1″ from B.

34. (*b*) Draw the top and front views of a regular triangular pyramid standing on a plane through A(—$1\frac{3}{4}$, $2\frac{1}{4}$, —$3\frac{1}{2}$) parallel to the lines B(—$3\frac{1}{2}$, —$\frac{3}{4}$, —$2\frac{1}{4}$) C($2\frac{1}{4}$, $3\frac{1}{4}$, —4) and E(—$1\frac{1}{4}$, $3\frac{1}{2}$, —$2\frac{1}{2}$) F($1\frac{3}{4}$, 0, —$5\frac{3}{4}$). One side of the base lies in the line of intersection of the plane of the base and a plane through A perpendicular to the line BC. The vertex is at the point O($2\frac{3}{4}$, —$\frac{3}{4}$, —$\frac{3}{4}$). Find the points in which BC and EF pierce the pyramid.

The above problem can be varied by making the base a square, regular pentagon, regular hexagon, or changing the pyramid to a cone with the base tangent to the line of intersection of the two planes.

35. (*b*) C(—$1\frac{1}{4}$, $2\frac{1}{4}$, —2) E($1\frac{1}{2}$, 4, —$4\frac{3}{4}$) is a diagonal of a cube. One corner of the cube is in a plane which makes 60° with H and contains a horizontal line passing through P(—$3\frac{3}{4}$, $\frac{1}{2}$, —$\frac{1}{2}$) and making 45° with V. Draw the top and front views of the cube.

36. (*b*) C($-3\frac{1}{4}$, $1\frac{1}{4}$, $-1\frac{1}{4}$) E($1\frac{1}{4}$, $1\frac{1}{4}$, $-1\frac{1}{4}$) is a diagonal of a cube which has one corner in a horizontal plane through the origin. Draw the top and front views of the cube.

37. (*b*) One edge of a $2\frac{1}{2}''$ cube lies along the line A($-2\frac{1}{4}$, $3\frac{1}{4}$, -4) B($1\frac{1}{4}$, 0, $-2\frac{3}{4}$) and an adjacent edge passes through the point P ($\frac{1}{2}$, $1\frac{1}{2}$, ?) in such a direction that its top view makes 60° with the top view of AB. Draw the top and front views of the cube.

38. (*b*) The face of the cube in problem 37 which lies on the plane ABP is the base of a right square pyramid. Draw the top and front views of the pyramid when it has an altitude of $3\frac{1}{2}''$.

39. (*b*) The common perpendicular to the lines A(-2, 4, $-1\frac{1}{2}$) B($-\frac{3}{4}$, $\frac{1}{2}$, $-4\frac{1}{4}$) and C($-3\frac{3}{4}$, $2\frac{1}{2}$, $-2\frac{1}{2}$) E($-\frac{3}{4}$, $3\frac{1}{4}$, $-4\frac{1}{2}$) is the axis of a right rectangular prism. The diagonals of the upper and lower bases coincide with AB and CE respectively. The shorter edge of the base is $1\frac{3}{4}''$ long. Draw the top and front views of the prism.

40. (*b*) Substitute A(2, $\frac{1}{2}$, $-2\frac{1}{4}$) B(2, $2\frac{1}{2}$, $-4\frac{1}{4}$) and C($-3\frac{1}{2}$, $2\frac{3}{4}$, -2) E($\frac{1}{4}$, $5\frac{1}{2}$, $-\frac{1}{2}$) in problem 39.

41. (*b*) Draw the top and front views of a line which passes through C($-1\frac{1}{2}$, $1\frac{1}{2}$, -3), intersects the line A($-\frac{1}{2}$, 4, $-1\frac{1}{2}$) B(2, $2\frac{1}{2}$ -5), and passes $1\frac{1}{2}''$ from the point E($2\frac{1}{4}$, 4, -2).

42. (*b*) Find the top and front views of the shortest line which is parallel to H and terminates in the lines A($-1\frac{1}{2}$, 4, $-2\frac{1}{4}$) B($-\frac{3}{4}$, $1\frac{3}{4}$, $-3\frac{3}{4}$) and C($-1\frac{1}{8}$, $1\frac{5}{8}$, $-2\frac{1}{4}$) E($-2\frac{3}{4}$, $3\frac{7}{8}$, -3).

43. (*b*) A 1″ square stick has its edges making 30° to V and oblique to H. Find the true size of the section of this stick by a plane parallel to V; also by a plane parallel to P.

44. (*b*) Draw the top and front views of a triangular pyramid O—ABC. The base is A($-\frac{3}{4}$, $3\frac{1}{4}$, -5) B($1\frac{1}{4}$, $1\frac{7}{8}$, $-3\frac{1}{4}$) C($3\frac{1}{8}$, $4\frac{1}{2}$, $-4\frac{1}{4}$). The altitude is $2\frac{3}{4}''$, while OA=$3\frac{1}{2}''$ and OB=3″ are two of the lateral edges.

45. (*b*) Draw the top and front views of a tetrahedron two edges of which are 3″, two 4″, and two 5″. The edges are to be oblique to H and V.

46. (*b*) Draw the top and front views of three spheres, each tangent to the other two and having the following radii: $1\frac{1}{2}''$, 2″, $2\frac{1}{4}''$. Draw the views of a fourth sphere with radius 1″ which will be tangent to the other three spheres. The plane of the centers of any three spheres is to be oblique to both H and V.

47. (*a*) Given an acute triangle on a horizontal plane. Find a point in space which joined with the vertices of the triangle will form a trirectangular tetrahedron.

48. (*b*) Draw the top and front views of a 3″ cube with one corner at O(0, 3, -2) and three edges along the lines joining O with the points A(-2, $2\frac{1}{8}$, ?), B($1\frac{3}{4}$, $5\frac{7}{8}$, ?), and C($2\frac{1}{2}$, $\frac{7}{8}$, ?).

49. (*b*) Draw the top and front views of an equilateral triangle with one corner at A($-\frac{1}{4}$, $3\frac{7}{8}$, $-4\frac{3}{4}$), another at B($1\frac{3}{4}$, $2\frac{1}{8}$, $-2\frac{5}{8}$), and a side along the line joining B with P($3\frac{3}{4}$, $5\frac{1}{2}$, ?).

50. (*b*) Draw the top and front views of a cube with one corner at A(0, 3, —2), another at B(2, $1\frac{1}{4}$, $-2\frac{3}{4}$), and a side along the line joining A with C(—2, 2, ?).

51. (*b*) Given the points O($\frac{1}{2}$, $\frac{3}{4}$, $-1\frac{3}{4}$), A($2\frac{3}{4}$, 2, —3), B($-1\frac{1}{4}$, $1\frac{3}{4}$, $-3\frac{1}{8}$), C($-2\frac{1}{2}$, $3\frac{3}{4}$, —3), and E(—2, $5\frac{5}{8}$, $-5\frac{1}{8}$). Draw the top and front views of a parallelopiped with edges along the lines OA, OB, and OC, and a 5″ diagonal along OE.

52. (*a*) A ray of light passes through P(3, 3, —2) and is reflected from V at the point O(1, 0, $-\frac{3}{4}$). Where will the ray strike H?

53. (*a*) Find the top and front views of a ray of light which emanates from A(—2, 3, —2) and after being reflected from H and V passes through B(2, 2, —1).

54. (*b*) A ray of light from A(—3, $2\frac{1}{2}$, $-3\frac{3}{4}$) is reflected at B(0, $3\frac{3}{4}$, $-2\frac{3}{8}$) to C(—1, $\frac{5}{8}$, $-4\frac{1}{8}$). Find the reflecting plane.

55. (*b*) Represent a plane which contains the line A($-1\frac{3}{4}$, 2, $-2\frac{1}{4}$) B(1, $3\frac{3}{4}$, $-3\frac{1}{2}$) and is in such a position that AB bisects the angle between a horizontal line and frontal line of the plane. Draw the horizontal and frontal lines through A.

56. (*a*) Draw the top, front, and left side views of the square with center at P($1\frac{3}{4}$, $2\frac{1}{2}$, —3) and side along the line A($2\frac{3}{4}$, $1\frac{3}{4}$, $-4\frac{1}{2}$) B($2\frac{3}{4}$, 4, $-2\frac{1}{2}$).

57. (*a*) Draw three views of a squre with center at P(2, 2, —2) and side along the line A($3\frac{1}{4}$, 1, $-3\frac{1}{4}$) B($3\frac{1}{4}$, $3\frac{1}{2}$, $-1\frac{1}{2}$).

58. (*a*) A 2″ x $2\frac{1}{2}$″ rectangle has its center on the line O($\frac{1}{4}$, $\frac{1}{2}$, $-4\frac{1}{2}$) P(2, $3\frac{1}{4}$, —2) and lies in a plane which contains the line M($-\frac{1}{2}$, $\frac{1}{2}$, $-2\frac{3}{4}$) N(3, $2\frac{1}{4}$, $-2\frac{3}{4}$) and slopes downward to the back making 45° with H. Draw the top and front views of the rectangle when one side makes 30° with MN.

59. (*b*) Draw the top and front views of a $2\frac{1}{2}$″ cube with one edge along the line A($1\frac{1}{2}$, $3\frac{3}{4}$, $-1\frac{3}{4}$) B($1\frac{1}{2}$, $\frac{1}{4}$, $-5\frac{1}{4}$) and an adjacent edge passing through the point P(0, $3\frac{1}{4}$, $-3\frac{1}{4}$).

60. (*b*) Draw the top and front views of a $2\frac{1}{2}$″ cube standing on the plane A(0, $3\frac{1}{2}$, —2) B(0, 0, $-5\frac{1}{2}$) P($-1\frac{1}{2}$, 3, $-3\frac{1}{2}$). One side of the base is to make 30° with AB.

61. (*b*) S($-1\frac{3}{4}$, 5, $-3\frac{1}{2}$) S ($\frac{3}{8}$, $2\frac{7}{8}$, $-3\frac{1}{2}$) is one line of a plane which makes 30° with H. Draw the top and front views of a $2\frac{1}{2}$″ cube which has its base in the plane. One side of the base is to make 30° with a frontal line of the plane.

62. (*b*) Given a plane A(—4, $\frac{1}{2}$, 0) B(1, $\frac{1}{2}$, —5) C(1, $5\frac{1}{2}$, 0) and a point O($-3\frac{1}{4}$, 3, $-2\frac{1}{2}$). Draw the top and front views of a cube which has one

corner at O, its base in the plane ABC, and one corner of the base in the line AC.

63. (*b*) Given a plane A(-4, $-\frac{1}{4}$, $-\frac{3}{4}$) B(1, $-\frac{1}{4}$, $-5\frac{3}{4}$) C(1, $4\frac{3}{4}$, $-\frac{3}{4}$). Draw the top and front views of a $2\frac{1}{4}''$ cube which has its base in the given plane. One corner of the base is at M(-1, ?, $-1\frac{1}{4}$) and another corner of the base is 1″ lower than the line AC.

64. (*b*) A plane which makes 30° with H and slopes downward to the right has S($-4\frac{1}{2}$, $-\frac{1}{4}$, $-4\frac{3}{4}$) S(0, $4\frac{1}{4}$, $-4\frac{3}{4}$) for a horizontal. C($1\frac{1}{8}$, 2, $-3\frac{1}{4}$) is an upper corner of a cube which has its base in the given plane. One corner of the base is to be on the horizontal AB. Draw the top and front views of the cube.

65. (*b*) Given a plane A(-4, $-\frac{1}{4}$, $-\frac{3}{4}$) B(1, $-\frac{1}{4}$, $-5\frac{3}{4}$) C(1, $4\frac{3}{4}$, $-\frac{3}{4}$). Draw the top and front views of a $2\frac{1}{4}''$ cube which has its base in the given plane. One corner of the base is to be on the line AC at the point O(-1, ?, ?) and one edge of the base is to make 30° with the line AC.

66. (*b*) S($-1\frac{3}{4}$, 5, $-3\frac{1}{2}$) S($\frac{3}{8}$, $2\frac{7}{8}$, $-3\frac{1}{2}$) is one line of a plane which makes 30° with H. Draw the top and front views of a $2\frac{1}{2}''$ cube which has its base in the plane. One side of the base is to make 30° with MN.

67. (*b*) Given a point P($-1\frac{3}{4}$, $3\frac{5}{8}$, $-3\frac{7}{8}$) and a line A($-2\frac{1}{8}$, 2, -3) B($-\frac{1}{8}$, $2\frac{5}{8}$, $-4\frac{1}{8}$). Draw the top and front views of a cube having AB for one edge and the plane ABP for the plane of the base.

68. (*b*) A(-1, 2, ?) B($\frac{1}{4}$. 3, ?) is a line which makes 45° with H. This line is one edge of a cube. Draw the top and front views of the cube when none of the edges are parallel to either H or V.

69. (*b*) A(-1, 2, ?) B($\frac{1}{4}$, 3, ?) is a line which is $2\frac{1}{2}''$ long. This line is one edge of a cube. Draw the top and front views of the cube when none of the edges are parallel to either H or V.

70. (*b*) M(-3, $2\frac{1}{8}$, ?) N(-2, $3\frac{1}{8}$, ?) is a line which makes 30°, 45° or 60° with H. Draw the top and front views of a cube having MN for one edge. One corner of the base is to be $\frac{1}{2}''$ lower than M.

71. (*b*) One edge of the base of a $2\frac{1}{2}''$ cube lies along the line A($-2\frac{5}{8}$, $1\frac{3}{4}$, $-2\frac{3}{4}$) B($\frac{3}{8}$, $2\frac{7}{8}$, $-4\frac{1}{2}$) and an adjacent edge passes through the point P($-\frac{3}{4}$, 4, $-4\frac{1}{2}$). Draw the top and front views of the cube.

72. (*b*) A(-3, $2\frac{1}{8}$, ?) B($-1\frac{1}{2}$, $3\frac{1}{2}$, ?) is a line which makes 60° with H. Draw the top and front views of the right square pyramid having AB for an axis. One corner of the base is to be $\frac{3}{4}''$ farther back than B, side of base $2\frac{1}{2}''$.

73. (*b*) A regular triangular pyramid whose altitude is 5″ stands on a plane which contains S(-2, 4, $-4\frac{1}{4}$) S($2\frac{3}{4}$, $1\frac{1}{4}$, $-4\frac{1}{4}$) and slopes downward to the back making 45° with H. Represent the pyramid by its top and front views when one corner of the base lies in the line A($-2\frac{1}{4}$, 3, $-2\frac{1}{8}$) B($-\frac{3}{4}$, $1\frac{1}{2}$, $-5\frac{1}{2}$) and one side of the base lies along the line of intersec-

tion of the plane of the base and a plane perpendicular to AB at a point 1″ from B.

74. (*b*) One edge of the base of a right square pyramid lies along the line A$(1\frac{1}{2}, 3\frac{1}{2}, -2)$ B$(1\frac{1}{2}, 0, -5\frac{1}{2})$ and an adjacent edge passes through the point P$(0, 3, -3\frac{1}{2})$. Draw the top and front views of the pyramid with side of base $2\frac{1}{2}''$ and altitude $3\frac{1}{2}''$.

75. (*b*) Given a plane A$(-4, \frac{1}{2}, 0)$ B$(1, \frac{1}{2}, -5)$ C$(1, 5\frac{1}{2}, 0)$ and a point O$(-2, 4\frac{1}{4}, -3)$. Draw the top and front views of a right square pyramid with vertex at O and base in the plane ABC. The side of the base is to be $2\frac{1}{4}''$, and one corner of the base to be in the line AC.

76. (*b*) O$(-2, 4\frac{1}{4}, -3)$ is the vertex of a right square pyramid which has its base on the plane A$(-4, \frac{1}{2}, 0)$ B$(1, \frac{1}{2}, -5)$ C$(1, 5\frac{1}{2}, 0)$. One side of the base is to make 30° with the line AB. Draw the top and front views of the pyramid with side of base $2\frac{1}{4}''$.

77. (*b*) The base of a right pyramid whose altitude is $3\frac{1}{2}''$ is an equilateral triangle with one side along the line A$(-4\frac{3}{8}, \frac{1}{4}, -4\frac{1}{4})$ B$(\frac{1}{4}, 1\frac{1}{2}. -\frac{7}{8})$ and one corner on the line M$(-4\frac{3}{8}, 2\frac{3}{8}, -6\frac{3}{8})$ N$(\frac{1}{4}, 3\frac{5}{8}, -3)$. Draw the top and front views of the pyramid.

78. (*b*) Draw the top and front views of a right square pyramid with vertex at O$(-2\frac{1}{2}, 2\frac{3}{4}, -4\frac{1}{2})$ and base in the plane A$(-4\frac{1}{2}. -1, -1\frac{1}{2})$ B$(\frac{1}{2}, -1, -6\frac{1}{2})$ C$(\frac{1}{2}, 4, -1\frac{1}{2})$. One corner of the base is to be on the line AC, side of base $2\frac{3}{4}''$.

79. (*b*) P$(-1, 1\frac{5}{8}, -3\frac{3}{4})$ is the center of an equilateral triangle having one side along the line A$(-4\frac{3}{8}, 0, -5\frac{1}{2})$ B$(1\frac{1}{4}, 1\frac{1}{2}, -1\frac{3}{8})$. This triangle is the base of a right pyramid with an altitude of $3\frac{1}{2}''$. Draw the top and front views of the pyramid.

80. (*b*) Draw the top and front views of a right square pyramid which has its base on the plane A$(-4, -1, -2)$ B$(0, -1, -6)$ C$(0, 3, -2)$. One corner of the base is on the line AC, at the point O$(-1, ?, ?)$ and one side of the base makes 30° with the line AC. Side of base of pyramid $2\frac{1}{4}''$, altitude 3″.

81. (*b*) a$(-3, 2\frac{1}{8}. ?)$ b$(-1\frac{1}{2}, 3\frac{1}{2}, ?)$ is the top view of a line which is $3\frac{1}{2}''$ long. Draw the top and front views of the right square pyramid having AB for an axis. One side of the base is to make 30° with a frontal on the plane of the base. Side of base $2\frac{1}{2}''$.

82. (*b*) P$(-\frac{1}{2}, 3\frac{1}{2}, -2\frac{1}{2})$ is the vertex of a right square pyramid which has its base in the plane A$(-3, 1, -2\frac{1}{2})$ B$(\frac{7}{8}, 1, -4\frac{3}{4})$ C$(-3, 3\frac{1}{4}, -4\frac{3}{4})$. One side of the base is to make 30° with AB. Draw the top and front views of the pyramid when the side of base is equal to the altitude.

83. (*b*) A$(-1\frac{3}{4}, 1\frac{3}{4}, -3)$ is one corner of the base of a right square prism which has its axis along the line B$(-1\frac{1}{4}, \frac{1}{4}, -1\frac{3}{4})$ C$(2\frac{1}{2}. 4, ?)$. The axis makes 30° with H, and is 4″ long. Draw the top and front views of the prism.

CHAPTER II

APPLICATIONS OF THE ELEMENTARY PRINCIPLES OF THE POINT, LINE, AND PLANE

SHADES AND SHADOWS

69. In order that the views of an object shall more nearly represent the object as it appears, the shadows which the different parts of the object cast on other parts or on the planes of projection are sometimes shown. Shadows are used more in architecture than in any other branch of engineering but every engineer should know how to find the shadow of a simple object. Only the elementary principles of the subject will be outlined here.

70. Definitions. If an opaque object is placed between a surface and the source of light, the object will cast a shadow upon the surface. This is due to the fact that the object intercepts the rays of light and prevents their reaching the surface upon which the shadow is cast. When the light strikes an opaque body, that portion of the surface of the body which is turned away from the source of light and which, therefore, does not receive any of the light rays is said to be in the shade.

The source of light is the sun and because of its distance from the earth the rays of light are assumed to be parallel. For the sake of uniformity, the direction of a ray of light is the direction of the diagonal of a cube which joins the upper, left hand, front corner with the lower, right hand, back corner when the cube has two faces parallel to V and two parallel to H. In other words a ray of light is considered as coming over the left shoulder of the observer so that both its top and front views make angles of 45° with the ground line.

71. Shadows of points and lines. To find the shadow of a **point** on a surface, pass a ray of light through the point and find where this ray pierces the surface. This piercing point is the required shadow point.

To find the shadow of a **straight line** on a plane, find the shadows of two points of the line on the plane and join them by a straight line. If the shadow of the straight line falls upon two or more planes, find the shadows of two points of the line on each plane and join the corresponding shadow points. The shadow of the straight line is a broken line which changes direction on the line of intersection of the planes upon which the shadow is cast.

To find the shadow of a straight line on a curved surface, find the shadows of several points of the line and join these shadow points in the proper order.

To find the shadow of a **curved line** on a surface, it is usually necessary to find the shadows of several points of the curve on the surface and then join the shadow points in the proper order.

The shadow of a **plane figure** on a plane which is parallel to the plane of the figure is the same size and form as the figure itself.

72. To find the shadow of a square prism on the plane of the base.

Let the prism be given as shown in Fig. 48.

Analysis. Since the shadow of the prism is to be cast upon the plane of the base, it is evident that the base of the prism coincides with its own shadow. The shadow of the top is found by passing rays of light through the points A, B, C, and D and finding where these rays pierce the plane of the base. The straight lines joining these piercing points, in the proper order, will form the outline of the shadow of the top. The shadows of the corners of the upper base joined with the shadows of the corresponding corners of the lower base will be the shadows of the lateral edges (Art. 71). This completes the outline of the shadow of the prism.

Construction. bb_s is the top and $b'b'_s$ is the front view of a ray of light which passes through B. The front view b'_s of the point where this ray pierces the plane of the base is first found and then the top view b_s located. Then B_s is the shadow of the

point B upon the plane of the base of the prism. In like manner C_s, D_s, and A_s, are respectively the shadows cast by C, D, and A upon the plane of the base of the prism. Joining these

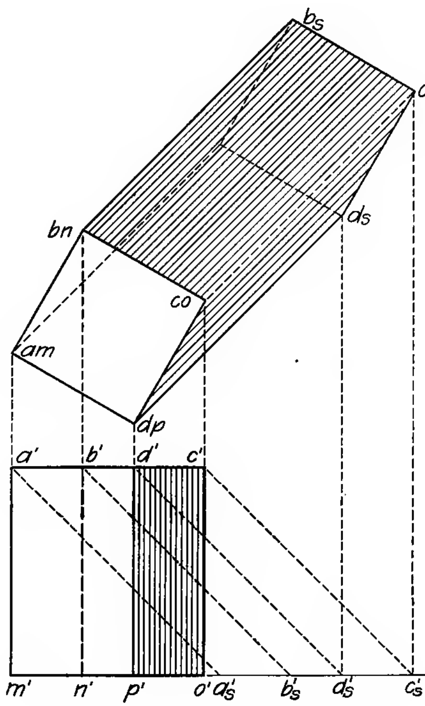


FIG. 48.—*Shadow of square prism.*

points in order gives the shadow of the top on the plane of the base. Joining B_s with N gives the shadow of the vertical edge BN. The shadows of the other vertical edges are found in the same manner. Then $nb_sc_sd_sp$ is the top view of the outline of the shadow of the prism on the plane of the base of the prism.

The face COPD of the object is turned away from the light and is therefore in the shade. This face is visible in the front view, and the fact that it is in the shade is indicated as shown in the figure.

From the figure, it is seen that a line and its shadow are parallel when the line is parallel to the plane upon which the shadow is cast.

73. To find the shadow of an object on the plane of the base and on the object itself.

Let the object be given as shown in Fig. 49.

This problem illustrates the finding of the shadow which an opaque object will cast upon itself as well as the plane of the base. It also shows the method of finding the shadow on a vertical plane in addition to finding the shadow on a horizontal plane.

Analysis. All of the edges shown are straight lines. If all of the shadow of any edge falls on one plane, it will be necessary to find the shadow of only two points of the edge and join them by a straight line. If the shadow of an edge falls on more than one plane, two shadow points should be found on each plane and joined by a straight line. The shadow of any point of an edge falls upon the plane which is first pierced by the ray of light passing through that point (Art. 71).

Construction. Each point of the top view of the shadow which the object casts upon the plane of the base is found as illustrated by points C and D, Fig. 48. For this shadow it is only necessary to find where the rays of light pierce a horizontal plane. The shadow which the upper part of the object casts upon the rectangular collar is found as illustrated by corner D, d'_s being first located then d_s.

The rectangular collar casts a shadow upon the vertical faces of the object. The horizontal edge BE of the collar casts part of its shadow, b_sc_s, on the plane of the base and the remainder, $c'_so'_se'_s$ on two of the vertical faces of the object. o_s and e_s are first located and then the front views o'_s and e'_s. To determine the part CO of the edge BE which casts a shadow on the vertical face, draw the top views of rays of light through c_s and o_s, thus locating the points c and o. The front view shows the shadow which the collar casts upon the vertical faces of the object.

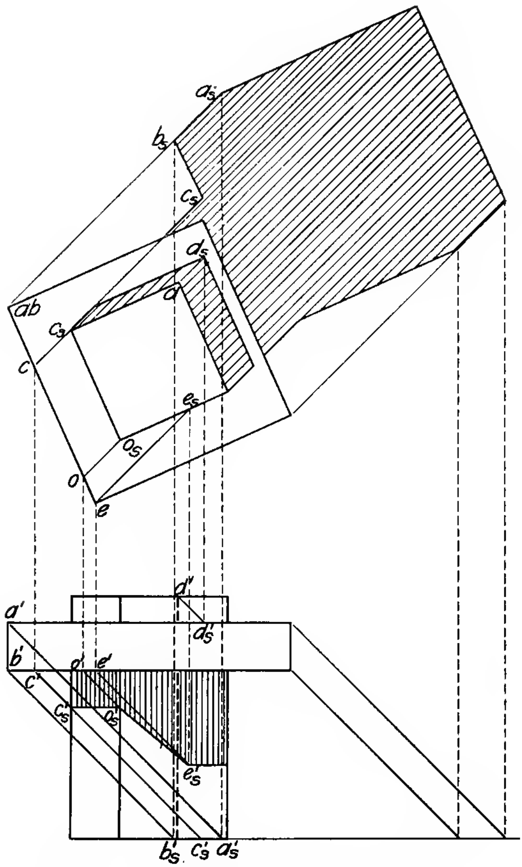


FIG. 49.—*Shadow of object on plane of base and object itself.*

74. To find the shadow of a cylindrical column and cap on a horizontal plane and also the shadow of the cap on the column.

Let the column and the cap be given as shown in Fig 50.

Analysis. Since the upper surface of the cap is a horizontal

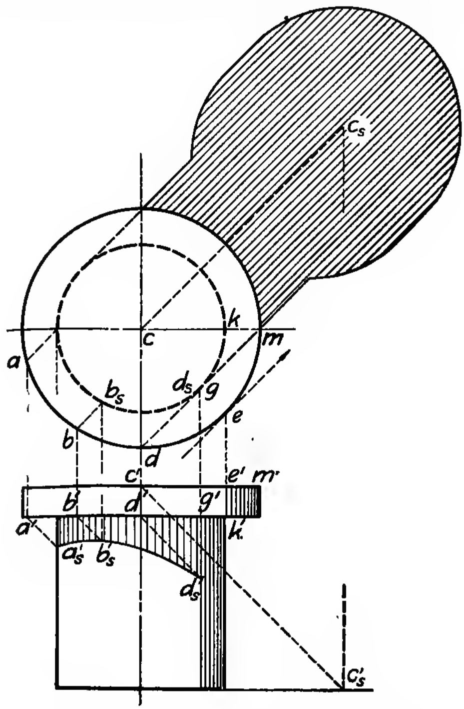


FIG. 50.—*Shadow of column and cap.*

circle, its shadow on a horizontal plane is a circle of the same radius as the cap. The shadow of the center of the circle is the center of the shadow of the circle. Finding the shadow of the center and drawing a circle with a radius equal to the radius of the cap, will give the shadow of the top of the cap on a horizontal plane. The shadow of the lower surface of the cap on a horizontal plane is found in the same way. Common tan-

gents to these circles drawn in the direction of the top view of a ray of light are the shadows of the extreme elements of the cap and complete the shadow of the cap on the horizontal plane. The shadow of the column on a horizontal plane is bounded by two lines drawn in the direction of the top view of a ray of light and tangent to the base of the column.

To find the shadow of the cap on the column, pass rays of light through points of the lower edge of the cap and find where they pierce the column. The top views b_s, d_s, etc., of these points are first determined and subsequently, the front views b_s', d_s', etc. A line joining these points is the boundary of the shadow of the cap on the column.

The side of the column and cap turned away from the source of light is in the shade. GK is the portion of the column and EM the portion of the cap which are respectively in the shade and visible in the front view.

75. Problems.

1. One edge of a 2″ cube lies on a horizontal plane and is oblique to V. The plane of one face of the cube makes 30° with H. Draw the top and front views of the cube and find its shadow on a horizontal plane.

2. Draw the top and front views of a 2″ cube when one diagonal is perpendicular to H. Find the shadow of the cube on a horizontal plane which passes through the lower end of the diagonal.

3. A right square pyramid has its vertex in a horizontal plane. The side of the base is 2″ The altitude is 3″ long, makes 60° with H, and is oblique to V. Draw the top and front views of the pyramid and find its shadow on a horizontal plane.

4. Draw the top and front views of a sphere which has a radius of $1\frac{1}{4}$″. Find at least twelve points in its shadow on a horizontal plane.

PLANE SECTIONS AND DEVELOPMENTS OF THE SURFACES OF PRISMS AND PYRAMIDS

76. By the development of a given surface, such as the surface of a prism or cone, is meant a plane area of such a form and size that it can be rolled or folded into the form of the given surface.

To find the line of intersection of a right prism with a given oblique plane and to develop the surface of the prism.

Let the prism be given as in Fig. 51, and let SSF represent the oblique plane.

Analysis. Take an auxiliary view of the prism and plane by looking in the direction of the horizontal line SS of the plane. In this view the plane is represented as one line and the points in which the lateral edges of the prism pierce the plane are apparent.

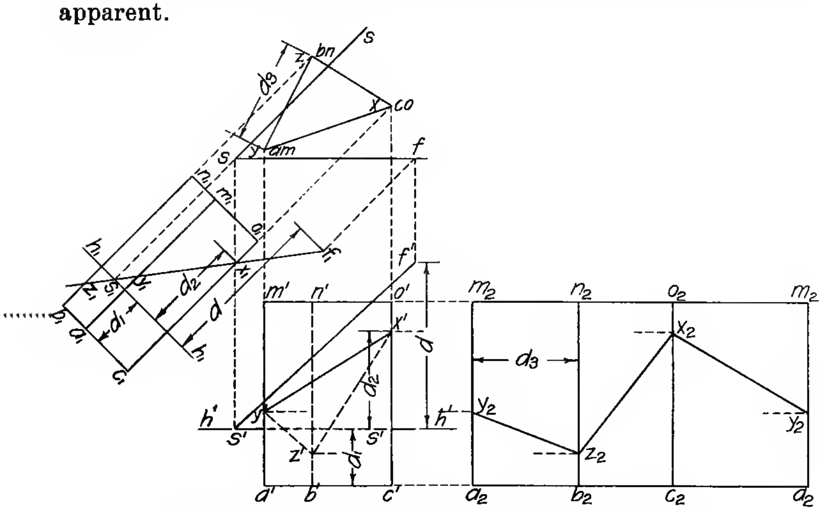


FIG. 51.—*Plane section and development of right prism.*

The development is made from the true size of the base of the prism and from the true lengths of its lateral edges. The true size of the base appears in top view while the true lengths of the lateral edges appear in both the front and auxiliary views.

Construction. In the auxiliary view, h_1h_1 is drawn at right angles to ss and at any convenient distance from the top view. h_1h_1 represents a horizontal plane at the level of SS. The lower base of the prism is the distance d_1 below this plane. s_1 is the auxiliary view of the horizontal line SS and f_1 represents the point F in this view. s_1f_1 represents the oblique plane and x_1, y_1, z_1, its intersection with the edges of the prism. The top and front views of X, Y, and Z are found from their auxiliary views.

It is best to place the development directly opposite the front view so that the true lengths of the elements can be projected across from that view. Draw a_2a_2 and m_2m_2 as extensions of $a'c'$ and $m'o'$ respectively. Lay off $a_2b_2=ab$, $b_2c_2=bc$, etc. Then $a_2m_2m_2a_2$ is the development of the lateral surface of the prism. Projecting the points x', y', and z' across to the proper edges determines the points x_2, y_2, and z_2, and these joined by lines represent the line of intersection on the development.

This method also applies to the plane section and development of a right cylinder.

Note. All the problems in plane sections of prisms and pyramids can be multiplied by changing the number of lateral faces of the surface.

When desired, the bases of the surfaces can be shown true size on the development. In this text the bases are omitted. Frequently the bases are shown true size in either the top or front view.

77. To develop the lateral surface of an oblique prism. Let the prism be given with its base in a horizontal plane and its lateral edges oblique to the plane of the base as show in Fig. 52.

Analysis. Take a section of the prism by a plane at right angles to the lateral edges. This section becomes a straight line on the development and the width of the faces from one lateral edge to the next can be measured along it. The lateral edges are drawn at right angles to this section line in the develop-

ment, and the distances from the plane of the section to the upper and lower bases are laid off on the respective edges in the development. Straight lines joining in the proper order the ends of the lateral edges complete the development of the prism.

Construction. From the top and front views of the prism an auxiliary view is drawn. This view is taken from a direction

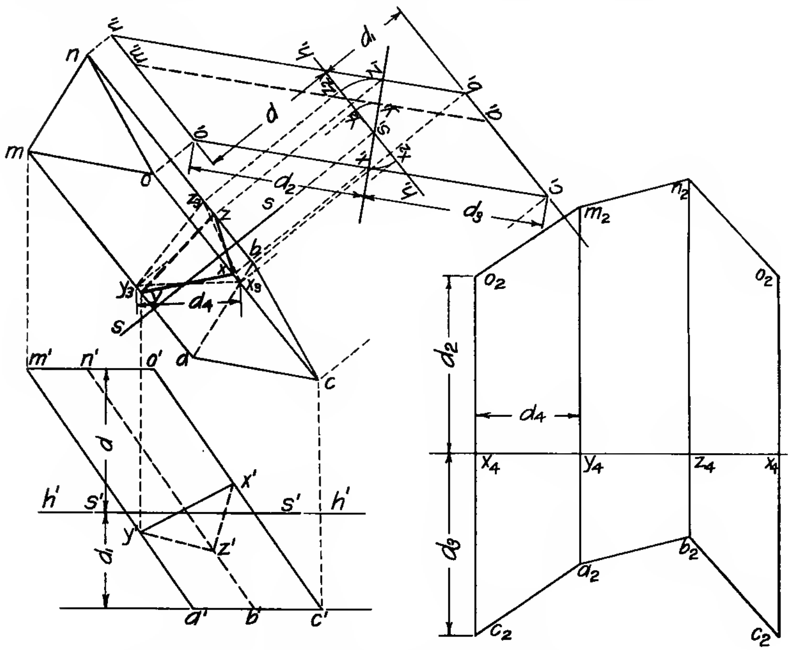


FIG. 52.—*Plane section and development of oblique prism.*

such that the lateral edges of the prism show true length, as c_1o_1, a_1m_1, b_1n_1. The plane which cuts the prism at right angles to the edges appearsin the auxiliary view as a line at right angles to the edges and cutting them in the points x_1, y_1, and z_1. This section can be taken anywhere between the bases. x, y, z is the top and x', y', z' the front view of this section. Any point, as s_1, in the auxiliary view of the cutting plane can be

taken as the view of a horizontal axis SS. When the triangle XYZ is revolved into a horizontal position about SS as an axis, the top view x_3, y_3, z_3 shows its true size. On any straight line, as x_4x_4, lay off $x_4y_4 = x_3y_3$, $y_4z_4 = y_3z_3$, etc. At the points x_4, y_4, z_4, x_4 erect perpendiculars to x_4x_4 and measure off on them the respective lengths of the edges obtained from the auxiliary view. The distance d_2 locates the point O in the development at o_2 and d_3 locates C at c_2. The ends of the other edges are located in a similar manner and when these ends are joined in the proper order the development is completed.

78. To develop the lateral surface of an oblique pyramid and to show on the development the line of intersection of the surface with an oblique plane.

Let the pyramid be given with its base in a horizontal plane Fig. 53, and let SSF be the oblique plane cutting the lateral surface.

Analysis. Find the points where the lateral edges of the pyramid pierce the oblique plane and join these piercing points in proper order with straight lines. This gives the intersection of the surface with the oblique plane. Lay out the faces of the pyramid on the development, constructing each face as a triangle having given the three sides. The line of intersection with the oblique plane is laid off on the development by measuring along each edge the true distance from the vertex to the point in which this edge is cut by the oblique plane. Joining these points by straight lines, gives the intersection on the development.

Construction. The auxiliary view is taken by looking in the direction of the line SS. In this view, s_1f_1 represents the oblique plane cutting the lateral edges in the points x_1, y_1, and z_1. xyz is the top and $x'y'z'$ the front view of the intersection. The top view shows the base ABC in its true size. If each of the lateral edges of the pyramid be rotated about a vertical axis through the vertex until the edge comes parallel to V, the front view will show its true length. For example, in the top view c moves in a circle with center at v until it reaches c_2, while in the front view c' moves in a line parallel to G. L. until it

reaches c'_2. Then $v'c'_2$ is the true length of the edge VC. In the front view, the point z', on the line $v'c'$, moves parallel to G. L. until it reaches z_2'. $v'z'_2$ is the true length of the line VZ. In the development each triangle as $v_3a_3c_3$, $v_3c_3b_3$, etc., is constructed by arcs, the length of three sides of the triangle being given. The true distances from the vertex to the points in which the oblique plane cuts the lateral edges are taken from the front view and laid off on the development, locating the line of intersection $x_3z_3y_3x_3$.

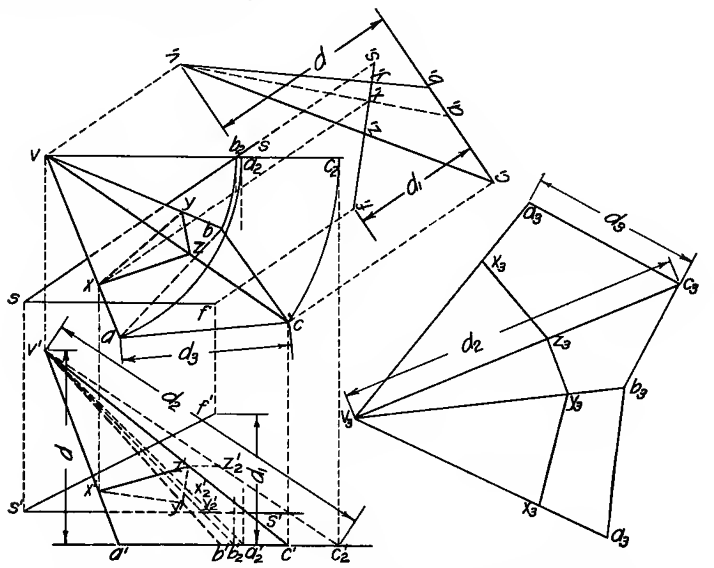


FIG. 53.—*Plane section and development of oblique pyramid.*

In a regular pyramid, the development is a series of isosceles triangles, with the equal sides the length of the laterial edges of the pyramid and the bases the length of the sides of the base of the pyramid.

79. Problems.

Use a 7″x10″ rectangle for each of the following problems.

1. Assume any four points which do not lie in a plane, as A($-1\frac{3}{4}$, 1, -4), B($-1\frac{1}{4}$, $2\frac{1}{2}$, $-2\frac{3}{4}$), C($-\frac{1}{4}$, $1\frac{1}{4}$, $-3\frac{1}{2}$), and D (1, 2, $-3\frac{1}{2}$), and draw through them four parallel lines in such a way that the plane section of the prism which has the four lines for edges will be a parallelogram.

2. Given a tetrahedron, O(0, 3, 0) A(-2, $2\frac{1}{2}$, -1) B(-3, 1, -3) C(-1, $\frac{1}{4}$, $-2\frac{3}{4}$). Select a point on one of the edges and then draw the top and front views of the shortest path to be followed on the surface of the tetrahedron, beginning at this point, crossing three of the faces and returning to the same point.

3. Given two planes, A($\frac{1}{8}$, $1\frac{1}{2}$, 0) B($-2\frac{1}{2}$, 0, 0) C($-1\frac{1}{2}$, 0, $-2\frac{1}{2}$) and A($\frac{1}{8}$, $1\frac{1}{2}$, 0) E(1, 0, 0) C($-1\frac{1}{2}$, 0, $-2\frac{1}{2}$). Draw the top and front views of the shortest path on the planes from B to E.

4. Pass a plane so as to cut a regular hexagon from a 3″ cube.

5. Cut a parallelogram having one side 2″ long from the pyramid O—ABCE. O($1\frac{1}{2}$, $3\frac{1}{2}$, 0) A($-2\frac{1}{8}$, $\frac{1}{2}$, -4) B($\frac{1}{2}$, $-\frac{1}{2}$, -4) C($2\frac{1}{2}$, $1\frac{1}{2}$, -4) E($-1\frac{3}{8}$, $3\frac{1}{2}$, -4.)

6. Given a square prism with its axis perpendicular to H. Side of base $2\frac{1}{4}$″. Cut a parallelogram from the prism having one side 3″ long and one angle 60°.

7. Given a square prism with its axis perpendicular to H. Side of base $2\frac{1}{4}$″. Cut a parallelogram from the prism having one diagonal $6\frac{1}{2}$″ long and one angle 60°.

Use a 10″ x 14″ rectangle for each of the following problems. The base line should be drawn through the center of the rectangle and parallel to the longer side.

8. A triangle A($-6\frac{3}{4}$, $2\frac{5}{32}$, $-4\frac{3}{4}$) B($-5\frac{1}{2}$, 0, $-4\frac{3}{4}$) C($-4\frac{1}{4}$, $2\frac{5}{32}$, $-4\frac{3}{4}$) is the base of an oblique prism. The top view of the edges make 45° with V. Thee dges are $4\frac{1}{2}$″ long and make 60° with H. Develop the surface of the prism.

9. The base of a triangular pyramid is A(-6, $2\frac{1}{4}$, -1) B($-3\frac{3}{4}$, $1\frac{1}{2}$, -1) C($-4\frac{1}{2}$, $4\frac{3}{4}$, -1). The vertex is V($-2\frac{1}{2}$, 4, ?). The axis passes through the center of the circle ABC and makes 60° with H. A plane passes through P($-2\frac{3}{4}$, $1\frac{1}{4}$, -2) and is perpendicular to the axis. Find the line of intersection of the plane and the pyramid and develop the pyramid, showing the line of intersection on the development.

INTERSECTIONS OF THE SURFACES OF PRISMS AND PYRAMIDS

80. General method. Find where the edges of each solid pierces the faces of the other solid. A series of straight lines joining in order these piercing points is the required line of intersection.

The construction for such a problem can be made by the method of finding where a line pierces a plane. If, however, one of the surfaces is a prism, the best method for finding the intersection is to take an auxiliary view looking in the direction of the axis of the prism. This view will show the faces of the prism as lines and the intersections of these faces with the edges of the other surface are readily found. This method is particularly desirable when the axis of the prism is parallel to H or V. If the axis is oblique to both H and V, a second auxiliary view will be necessary to make the faces of the prism appear as lines.

81. To find the line of intersection of the surfaces of a prism and a pyramid by the auxiliary view method.

Let the prism and pyramid be given as in Fig. 54.

Analysis. The auxiliary view in the direction of the axis shows the prism as a square. From this view, the points in which the edges of the pyramid pierce the faces of the prism are found by inspection. To find the points in which an edge of the prism pierces the pyramid, use an auxiliary plane containing this edge and parallel to the base of the pyramid. This plane cuts lines from the faces of the pyramid which intersect the edge of the prism in the required points.

Construction. In the auxiliary view, the points in which the lateral edges of the pyramid pierce the faces of the prism are seen to be x_1, y_1, w_1, etc. The top views x, y, w, etc., of these points are found from the auxiliary view by projection and then the front views x', y', w', etc., are found from the top views. h_1h_1 is the auxiliary view of a plane which is parallel to the

base of the pyramid and contains the edge PP of the prism. This plane cuts a square from the pyramid which has one corner at R and sides parallel to the sides of the base of the pyramid. The square cuts the edge PP of the prism in the required points Q and Z. Joining the points thus found in the proper order gives the required line of intersection.

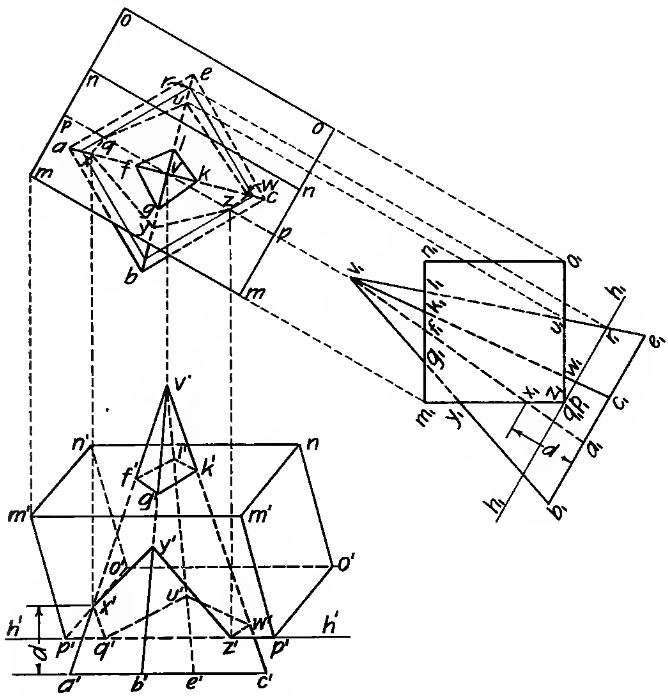


FIG 51.—*Intersection of prism and pyramid.*

82. Problems.

1. Find the line of intersection of a cube with a square prism when the axis of the prism coincides with the diagonal of the cube. Edge of cube $2\frac{1}{2}''$; side of base of prism $2\frac{1}{4}''$; altitude $5''$. Axis of prism is perpendicular to H, the plane of one face makes 15° with V. The H projection of one top edge of the cube is perpendicular to a face of the prism.

Develop the surface of the cube showing the line of intersection.

2. The same as problem 1 except that the prism has an equilateral triangle with side $2\frac{1}{2}''$ for a base.

3. The same as problem 1 except that the diagonal of the cube is $\frac{1}{2}''$ from the axis of the prism.

4. Find the line of intersection of a square prism and a right square pyramid. Prism $2''\text{x}2''\text{x}4\frac{1}{2}''$, with edges parallel to H and 30° with V. The plane of one face makes 30° with H and the lowest edge is $\frac{3}{4}''$ above the base of the pyramid. The base of the pyramid is a $2\frac{1}{2}''$ square resting on a horizontal plane, side of base 60° with V. The axis of the pyramid is perpendicular to H and is $5''$ long. The axes of the prism and the pyramid are $\frac{1}{8}''$ apart.

Note.—This problem can be multiplied by changing the number of faces of the prism or pyramid, by changing the distance between the axes, by changing the angle which the plane of the face of the prism makes with H, or by raising or lowering the prism.

CHAPTER III

CURVED LINES AND SURFACES

GENERATION AND CLASSIFICATION OF LINES

83. A line is the path of a moving point.

Lines are of two general classes:

I. **A straight line** or **right line** is the path of a point which moves always in the same direction.

II. **A curved line** is the path of a point which moves so as to change its direction continually.

Curved lines are of two kinds:

(A) **A curve of single curvature** or **plane curve** is a curve in which all positions of the moving point line in one plane. Examples, circle, ellipse.

(B) **A curve of double curvature** or **space curve** is a curve in which four positions of the moving point do not lie in one plane, where these four points are taken one right after the other along the curve. Such points are frequently called "consecutive points." Example, the edge of an ordinary screw thread. The line of intersection of two curved surfaces is usually of this form.

OUTLINE

- Lines
  - I Straight or Right Lines
  - II Curved Lines
    - (A) Curves of Single Curvature or Plane Curves
      - Circle
      - Ellipse
      - Parabola
      - Hyperbola
      - Sinusoid
      - Cycloid
    - (B) Curves of Double Curvature or Space Curves
      - Helix

VIEWS OF A CURVE

84. The top view of a curve is the line joining the points in which the perpendiculars to H from points of the curve pierce a horizontal plane. Similarly, any other view is obtained by perpendiculars to the picture plane from points of the curve. Enough points must be determined in the views to give the desired degree of accuracy. If the form is known, the views of the curve can sometimes be constructed by determining only a few points, such as the view of a circle or ellipse.

Two views of a curve, on planes which are not parallel to each other, will usually determine the form of the curve in space. All views of a curve of double curvature are curved lines, but views of a plane curve from certain directions are straight lines. Answer the following questions regarding the views of a plane curve:

(a) In what position must the plane of the curve be placed in order that the top view of the curve be a straight line? In this position, how may the front view appear?

(b) How must the plane of the curve be placed in order to have the front view of the curve a straight line? Describe the top view of the curve.

(c) Can both top and front views of a curve be straight lines? Which view of the curve will show its true size?

(d) In general, what must be the direction of the line of sight to show a plane curve as a straight line? To show it in its true size?

TANGENTS AND NORMALS TO LINES

85. If in a secant line AB., Fig. 55, the point A be kept fixed and the point B moved along the curve until it coincides with A, **the secant AB becomes a tangent to the curve at the point A.**

Two curves are said to be tangent to each other at a point when they have **a common tangent at that point.**

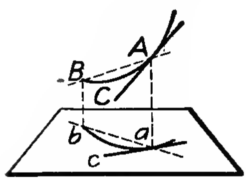


FIG. 55.—*Tangent to a curve.*

If a straight line is tangent to a plane curve, the tangent will lie in the plane of the curve. This is evident since the secant through the point of tangency is in the plane of the curve, and it remains in this plane as it moves to the position of the tangent.

86. If two lines are tangent to each other in space, their projections on the same plane will be tangent. Fig. 55 represents a curve in space with its secant AB. Let these lines be projected upon any plane. The projections of the points A and B will approach each other as the points A and B in space approach each other. When the secant AB becomes a tangent in space, the points A and B coincide and their projections will also coincide in the only point common to the projections of the straight line and curve. The projections or views of the lines are therefore tangent to each other at this common point.

If a straight line is tangent to a curve in space, its top view is tangent to the top view of the curve and its front view tangent to the front view of the curve. The top and front views of the point of tangency must lie in a line at right angles to the ground line.

87. Normals. If a line be drawn perpendicular to the tangent at the point of tangency, it is called a normal to the curve. There are an infinite number of normals to a curve at a point on the curve. However, the normal to a plane curve is understood to be the one lying in the plane of the curve unless otherwise stated.

88. Rectification of curves. To rectify a curve means to find a straight line equal in length to the curve. This is accomplished approximately by dividing the curve into a number of small arcs, so small that for all practical purposes the chords of these arcs may be taken as equal in length to the arcs themselves. The small chords are then laid off one after another along a straight line. This can be conveniently done with the dividers. The part of the line thus covered is the rectified curve.

CURVES OF SINGLE CURVATURE

89. The most common plane curves are obtained by cutting a right circular cone by a plane. These curves are the circle, ellipse, parabola, and hyperbola. Fig. 56 shows the position which the cutting plane must take with reference to the plane of the base of the cone in order to cut the particular curve desired. ϕ must be less than Θ and α greater than Θ.

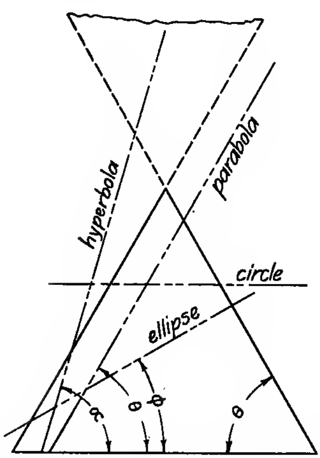


FIG. 56.—*Plane sections of a cone of revolution.*

A circle is the path of a point moving in a plane so that its distance from a given point is constant.

90. An ellipse is the path of a point moving in a plane so that the sum of its distances from two fixed points is constant.

Construction. Let F and F_1, Fig. 57, be the two fixed points and let the sum of the distances from these points to any point on the curve be equal to the line AB. Select any point, as O, on the line AB. With F as center and AO as radius, strike an arc; also with F_1 as center and BO

as radius, strike an arc cutting the first arc in the points P and Q. These are points on the curve since $FP+F_1P=AB$ and $FQ+F_1Q=AB$. Selecting any other point on AB, as O_1, and going through the construction as before, two other points P_1 and Q_1 are located. By continuing this process enough points can be located so that the curve can be drawn.

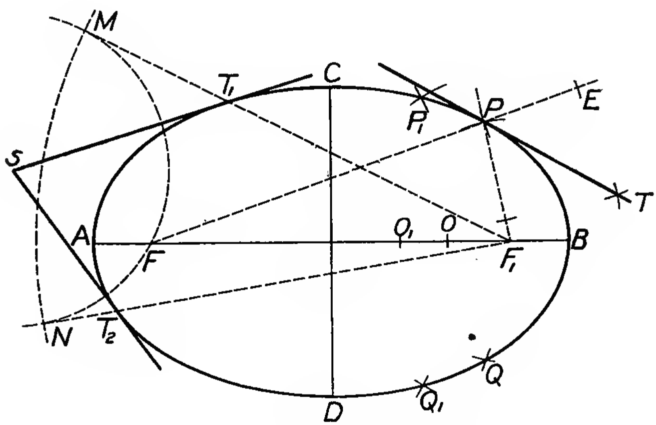


FIG. 57.—*Ellipse and tangents.*

The points F and F_1 are **foci** of the ellipse. AB is the **major axis**. The **minor axis** CD is perpendicular to the major axis at its middle point. The intersection of the axes is the **center** of the ellipse. Lines drawn from any point on the curve, as P, to the foci F and F_1 are called **focal radii**.

If the axes of an ellipse are given, the foci can be found by taking one-half the major axis as a radius and with the end of the minor axis as a center, cut the major axis in two points which are the foci.

There are many other ways of constructing an ellipse. One of the best is that known as the "trammel method." This

method is illustrated in Fig. 58, and is as follows: On a straight edge (the edge of a thin card or strip of paper is suitable for the purpose) lay off three points A, B, C, in such a way that AC is half the major and BC half the minor axis of the required ellipse. If the straight edge be moved in such a manner that the point A moves on the minor and B on the major axis, the point C will trace the ellipse.

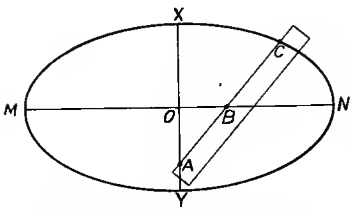


FIG. 58.— *Trammel method for constructing an ellipse.*

There are instruments, called ellipsographs, on the market which will trace an ellipse by the mere turning of a handle.

91. To draw a tangent to the ellipse at a point on the curve. Let P, Fig. 57, be the point on the curve. Draw the focal radii PF and PF_1. The line PT which bisects the angle F_1PE is the required tangent. (Wood's Co-ordinate Geometry, Art. 108A.) This method applies to all conics.

To draw a tangent to the ellipse from a point without the curve. Let S, Fig. 57, be the point without the curve. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points M and N. F_1M and F_1N intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents. (Wood's Co-ordinate Geometry, Art. 108B.) This method applies to all conics.

92. A parabola is the path of a point moving in a plane so that its distance from a given point is always equal to its distance from a given straight line.

Construction. Let F, Fig. 59, be the given point and AB the given straight line. Draw a line CD from F perpendicular to AB. Through any point, as O, on CD, draw a line parallel with AB. With F as center and CO as radius strike an arc cutting this line in the two points P and Q. P and Q are points on the

parabola, since FP and FQ are each equal to the distance of P and Q from AB. Selecting some other point on the axis and going through a similar construction, two other points of the curve are located. By continuing this process, enough points can be located so that the curve can be drawn.

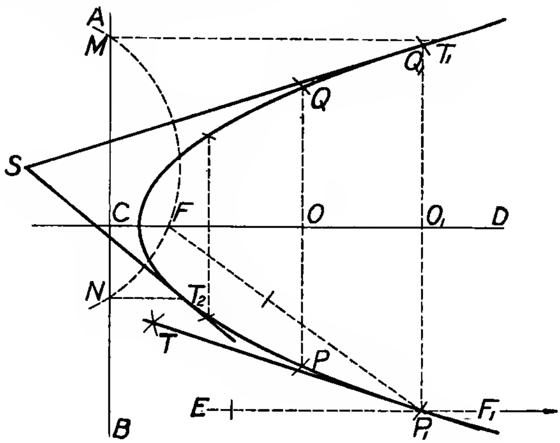


FIG. 59.—*Parabola and tangents.*

The point F is the **focus,** the line AB the **directrix**, and the line CD the **axis** of the parabola.

93. To draw a tangent to the parabola at a point on the curve. Let P_1, Fig. 59, be the point on the curve. Draw the focal radii FP_1 and F_1P_1. (To make this construction similar to that for the tangent to the ellipse, it is necessary to consider one focus F_1 of the parabola at an infinite distance on the axis). The line P_1T which bisects the angle FP_1E is the required tangent.

Another accurate and convenient method for constructing a tangent to a parabola at a point on the curve comes from the fact that the sub-tangent is bisected by the vertex of the curve. For example, extend the axis CD, Fig. 59, to the left and lay off on it from the vertex the distance from the vertex to O_1. This point joined with P_1 gives the tangent TP_1.

The normal to the curve at the point P_1 bisects the angle FP_1F_1. Therefore, a ray of light from F striking the parabola at any point as P_1 will be reflected parallel to the axis CD. For this reason, reflectors are made in the form of a surface generated by revolving the parabola about its axis CD.

To draw a tangent to the parabola from a point without the curve. Let S, Fig. 59, be the point without the curve. With S as center and SF as radius, strike an arc cutting the directrix in the points M and N. MF_1 and NF_1, parallel with the axis CD of the parabola, cut the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

94. A hyperbola is the path of a point moving in a plane so that the difference of its distances from two fixed points is constant.

Construction. Let F and F_1, Fig. 60, be the two fixed points and let the difference of the distances from these points to any

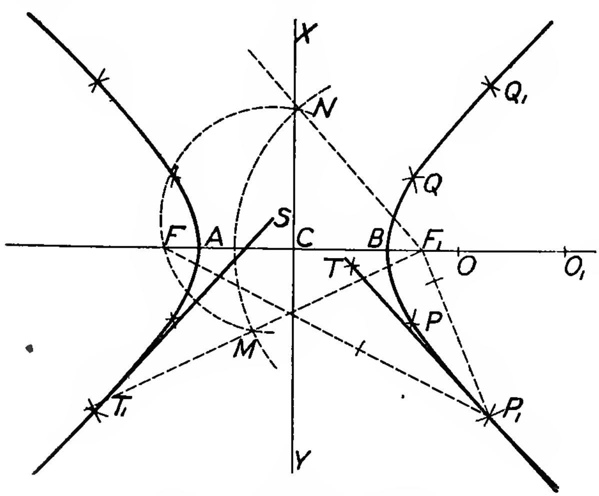


FIG. 60.—*Hyperbola and tangents.*

point on the curve be equal to the line AB. Select any point, as O, on the line AB extended. With F as center and AO as

radius, strike an arc; also with F_1 as center and BO as radius, strike an arc cutting the first arc in the points P and Q. These are points on the curve since $FP-F_1P=AB$ and $FQ-F_1Q=AB$. (If F_1 be taken as center and AO as radius and F as center and BO as radius, the points located will be on the other branch of the hyperbola). Selecting any other point on AB, as O_1, and going through the construction as before, two other points P_1 and Q_1 are located. By continuing this process enough points can be located so that the curve can be drawn.

The points F and F_1 are the **foci** of the hyperbola. The point C, midway between the foci F and F_1 is the **center** of the curve. The line AB is the **transverse axis.** The line XY, perpendicular to AB at its middle point, is the **indefinite conjugate axis** of the curve.

95. To draw a tangent to the hyperbola at a point on the curve. Let P_1, Fig. 60, be the point on the curve. Draw the focal radii FP_1 and F_1P_1. The line P_1T which bisects the angle F_1P_1F is the required tangent.

To draw a tangent to the hyperbola from a point without the curve. Let S, Fig. 60, be the point without the curve. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points M and N. F_1M and F_1N intersect the curve in the points of tangency. ST_1 is one of the tangents. The other point of tangency, where NF_1 intersects the curve, is without the limits of the drawing.

CURVES OF DOUBLE CURVATURE

96. An ordinary helix is the path of a point moving on the surface of the cylinder of revolution so as to intersect its elements at a constant acute angle.

The axis of the cylinder is the **axis** of the helix.

Construction. Let MN, Fig. 61, be the axis of the helix and P the generating point.

Suppose that for one complete turn around the axis the generating point moves through a vertical distance $m'n'$. This distance is the pitch of the helix.

Since the curve is on the surface of a cylinder of revolution a view of it looking in the direction of the axis will be the circle *pcfg*.

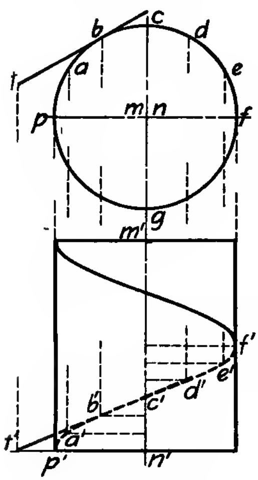


FIG. 61.—*Helix and tangent.*

To draw the view looking at right angles to the axis of the helix, in this case the front view, divide the circle *pcfg* into any number of equal parts, as twelve, and $m'n'$ into the same number of equal parts. Draw horizontal lines through the points of division of $m'n'$. Since the motions of the point around and along the axis are both uniform, the point in making one-twelfth of a complete turn around the axis will rise one-twelfth of the distance $m'n'$. a is the top and a' the front view of the point after making one-twelfth of a turn. In the same manner, the points b', c', d', etc., are found. The front view of the curve is drawn through the front views of these points.

Since the curve cuts all the elements of the cylinder at the same angle, it is evident that the helix will become a straight line if the cylinder is opened along an element and the surface rolled out into a plane. From this position it is seen that the

hypothenuse of a right triangle will form a helix if the altitude of the triangle coincides with an element of a right circular cylinder and the base of the triangle is wound around the base of the cylinder.

97. To construct a tangent to a helix at a point on the curve. Let B, Fig. 61, be the point at which the tangent is to be drawn. Let the triangle referred to in the above paragraph as being wound around the cylinder, be unrolled as far as the point of tangency B. The hypothenuse of the triangle in this position is tangent to the curve, since it touches the curve at the point B and makes the same angle with the horizontal plane as the curve. The base *bt* of the triangle is the top view of the tangent, and is equal in length to the arc *bp* of the circle *pcfg*. Therefore, to get the views of the tangent, draw its top view *bt* tangent to the circle *pcfg* at the point *b*. On this line lay off from *b* the true length of the arc *bap* of the circle. This will give the point *t* where the tangent pierces the plane of the base. The front view of this piercing point is at t' and this joined with b' gives the front view $b't'$ of the tangent.

GENERATION AND CLASSIFICATION OF SURFACES

98. A surface is the path of a moving line. An exception to this is the case of a straight line moving in the direction of its length.

The moving line is the **generatrix**, and its different positions are the **elements** of the surface.

Surfaces are of two general classes, ruled and double curved.

I. **A ruled surface** is one which can be generated by the motion of a straight line. A straight edge can therefore be placed against the surface in such a manner that it touches the surface throughout its length. Examples, cylinder, cone, helicoid.

II. **A double curved surface** is one which can only be generated by the motion of a curved line. Such a surface has no straight line elements. Examples, sphere, ellipsoid.

Ruled surfaces are of three kinds, plane, developable, and warped.

(A). A plane surface, or **a plane**, is a surface in which, if any two points are taken, the straight line joining these points lies wholly in the surface.

(B). A developable surface is a ruled surface in which any two straight line elements, coming one right after the other on the surface, lie in a plane. The surface can therefore be rolled into a plane or developed without undergoing distortion. Examples, cylinder, cone.

(C) A warped surface is a ruled surface in which any two straight line elements, coming one right after the other on the surface, do not lie in a plane. The surface cannot be developed without undergoing distortion. Example, the surface of a screw thread.

Developable surfaces are of three kinds, cylinders, cones, and convolutes.

I. **A cylinder** is a surface in which the rectilinear elements are parallel.

II. **A cone** is a surface in which the rectilinear elements intersect in a point.

III. **A convolute** is a surface in which the rectilinear elements are tangent to a curve of double curvature. Example, the surface having tangents to a helix for its elements.

OUTLINE

- Surfaces
  - I Ruled
    - (A) Plane
    - (B) Developable
      - I Cylinder
      - II Cone
      - III Convolute
    - (C) Warped
      - Helicoid
      - Hyperbolic Paraboloid
      - Conoid
      - Cylindroid
      - Hyperboloid of Revolution of one Sheet
  - II Double Curved
    - Sphere
    - Torus or Anchor Ring
    - Ellipsoid of Revolution
    - Paraboloid of Revolution
    - Hyperboloid of Revolution of two Sheets

SURFACES OF REVOLUTION

99. A surface of revolution is the path of a line *revolving* about a straight line as an axis.

It is evident that the intersection of this surface with a plane perpendicular to the axis is a circle. A surface of revolution may be generated by a circle having its center moving along a straight line, its different positions in parallel planes, and its radius changing according to a given law.

The interesection of the surface with a plane containing the axis is a **meridian line,** and the plane is a **meridian plane**. All

meridian lines of the same surface are identical. Any surface of revolution may be generated by revolving its meridian line about its axis.

OUTLINE

- Surfaces of Revolution
  - Ruled
    - Developable
      - Right circular cylinder
      - Right circular cone
    - Warped
      - Hyperboloid of Revolution of one Sheet
  - Double curved
    - Sphere
    - Torus
    - Ellipsoid of Revolution
      - Prolate
      - Oblate
    - Paraboloid of Revolution
    - Hyperboloid of Revolution of two Sheets

100. If two surfaces of revolution have a common axis, what will be their line of interection?

What must be the relative position of the axis and the generatrix to form a cyliuder of revolution? A cone of revolutiou? A hyperboloid of revolution of one sheet?

The cylinder and cone are the only single curved surfaces of revolution.

The hyperboloid of revolution of one sheet is the only warped surface of revolution.

TANGENT PLANES TO SURFACES, NORMAL LINES AND PLANES

101. Let AD_1 and AD_2, Fig. 62, be any two intersecting curves on a surface, and BCD a curve which if moved along the curves AD_1 and AD_2 will generate the given surface. The curve BCD may vary in form as it moves. The secants AB, BC, and AC lie in one plane. As the curve BCD moves toward A, the points of intersection B and C will travel along the curves AD_1 and AD_2 until the secants AB and AC become the tangents AB_1 and AC_2 at the point A. The curve AD_3 is the position of the moving curve BCD when the points B and C reach A. In this position, the secant BC becomes the tangent B_3C_3. The tangents AB_1, AC_2, and B_3C_3 all lie in one plane which is called the **tangent plane** to the surface at the point A. The point A is the **point of contact.**

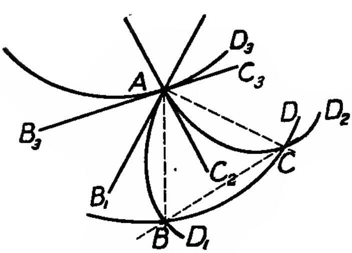


FIG. 62.—*Tangent plane to a surface.*

Since AD_1 and AD_2 are any curves of the surface, it follows that in general, the tangent at A to every curve of the surface through this point will lie in the tangent plane. Therefore, **the tangent plane will contain all straight lines tangent to lines of the surface at the point of contact.**

If any plane be passed through the point of contact, it will cut a straight line from the tangent plane and a line from the surface, and these lines will be tangent to each other.

Since two intersecting lines will determine a plane, it follows that **a plane tangent to a surface at a given point is determined by two straight lines tangent at this point to two lines of the surface.**

In any ruled surface, the rectilinear element through the point of contact lies in the tangent plane.

102. A tangent plane to a single curved surface is tangent all along a rectilinear element. In Fig. 63, let BD be the curve of the base of any single curved surface as a cylinder and AB and A_1B_1 any two rectilinear elements of the surface. From any point A of one element draw on the surface a curve AD_1 which cuts the other element at A_1. The chords AA_1 and BB_1 lie in the same plane and will intersect at some point as S_1. Now if the plane ABB_1A_1 be revolved about AB as an axis, the point A_1 will approach A along the curve AD_1 and B_1 will approach B along the curve BD. When the point A_1 reaches A, the secant AS_1 becomes the tangent AS to the curve AD_1, and at the same time B_1 reaches B, the secant S_1B becoming the tangent SB to the curve BD. The plane SAB is tangent to the cylinder at the point B (Art. 101), since it contains the tangent SB to the curve BD, and also the rectilinear element AB. This plane is also tangent to the cylinder at A, containing the tangent SA to the curve AD_1 and the element AB. Since A was taken as any point on the element AB, the plane SAB must be tangent to the surface at every point of this element.

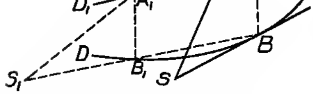


FIG. 63.—*Tangent plane to single curved surface.*

The same demonstration applies to the tangent plane to a cone.

In the case of the convolute, the element A_1B_1 is a curve instead of a straight line. The curve will change its form as the secant plane revolves, finally becoming a straight line coinciding with the axis AB. Otherwise the demonstration given above also applies to the convolute.

Since the plane is tangent all along the element, the intersection of the tangent plane with the plane of the base will be a straight line tangent to the curve which represents the base (Art. 101).

103. Tangent planes to warped surfaces. Since warped surfaces have straight line elements, the tangent plane to the surface at any point will contain the rectilinear element through that point (Art. 101). If any curve of the surface be drawn through the point of contact, the straight line tangent to the curve at this point and the rectilinear element through the point will determine the tangent plane (Art. 101).

If the warped surface is of such a form that there are two rectilinear elements through each point of the surface, then the tangent plane will be determined by the two elements through the point of contact.

Although the tangent plane to a warped surface contains a rectilinear element of the surface, it is, in general, tangent at only one point of this element. The tangent plane usually intersects the surface. These principles will be better understood when the tangent planes to some of the surfaces are drawn.

104. Tangent planes to double curved surfaces. Through the point on the surface at which the tangent plane is to be passed, draw any two curves of the surface. The straight lines tangent to these curves at their point of intersection will determine the tangent plane. If care is taken to select simple curves of the surface in simple positions with reference to the planes of projection, the tangents to these can be easily drawn.

In the case of a **double curved surface of revolution**, the simplest curves are usually the meridian curve and a circle which lies in a plane perpendicular to the axis of the surface.

105. Normals. A straight line perpendicular to the tangent plane at the point of contact is the **normal to the surface** at that point. Any plane containing a normal line is a **normal plane** to the surface.

SINGLE CURVED SURFACES

CYLINDERS

106. A cylinder is the path of a straight line which moves so as to touch a curved line and have all its positions parallel.

The curved line is the **directrix** and the moving line is the **rectilinear generatrix.**

The directrix can be a closed curve, as the circle or ellipse, or it can be a curve which is not closed, as the hyperbola or helix.

The intersection of the cylinder with a horizontal plane is usually taken as the **base,** although any plane section may be considered to be the base. If this base has a center, the straight line through it and parallel to the rectilinear elements is the **axis** of the cylinder.

A right cylinder is one in which the rectilinear elements are perpendicular to to the plane of the base. **A right circular cylinder** or **cylinder of revolution** is one formed by revolving a straight line about an axis to which it is parallel.

A section taken perpendicular to the elements of any cylinder is a **right section.** Cylinders are distinguished by the names of their right sections; as circular cylinders, elliptical cylinders, etc. If a cylinder is intersected by a plane parallel to the rectilinear generatrix it will cut rectilinear elements from the surface.

107. To represent the surface. A cylinder is usually represented by the views of its base and its extreme elements.

In Fig. 64, AECG is the base and EF the rectilinear generatrix. The extreme elements in the top view are tangent to the base and parallel to the top view of the rectilinear generatrix. The extreme elements in the front view are drawn through the extreme points of the front view of the base and parallel to the front view of the rectilinear generatrix. It will be noted in general that the extreme elements which are tangent to the base in the top view are not the extreme or "limiting" elements in the front view.

Since the surface is indefinite in extent, it can be represented

with the upper end broken off unless some definite portion of the surface is taken for a particular purpose.

108. To represent a rectilinear element of the surface. Through any point M of the base draw a straight line MN parallel to the rectiliner generatrix; this will be an element of the surface.

To represent a point of the surface. Locate a rectilinear element, as MN, and then take any point P of this element.

109. To represent a plane which contains a given point and is tangent to a cylinder.

The tangent plane to a cylinder is tangent all along a rectilinear element.

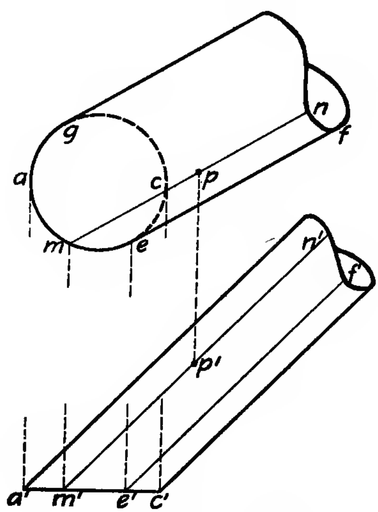


FIG. 64.—*Cylinder.*

The line of intersection of the tangent plane with the plane of the base is tangent to the curve of the base.

Let the cylinder be given as in Fig. 65, and let P be the given point.

Since the tangent plane must contain a rectilinear element of the cylinder, a straight line through P and parallel to the rec-

tilinear elements will lie in the tangent plane and will pierce the plane of the base at M. A tangent to the base of the cylinder through the point M is a horizontal line of the required tangent plane. An element of the cylinder through the point of tangency O will be the element of contact. This element, the horizontal line OM, and the line PM, or any two of these, will determine the tangent plane.

There can be as many tangent planes to the cylinder through the given point as there can be lines drawn tangent to the base of the cylinder through the point where the auxiliary line pierces the plane of the base.

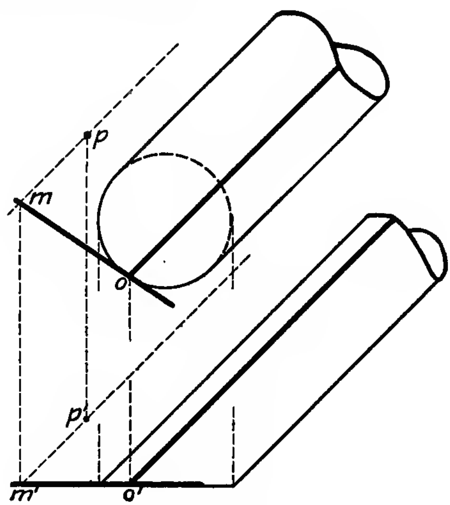


FIG. 65.—*Tangent plane to cylinder.*

110. Problems.

1. A right circular cylinder and a point are given by their top and front views. Represent a plane which contains the point and is tangent to the cylinder.

2. Represent a plane which is tangent to a cylinder at a given point of the surface.

3. Represent a plane which is tangent to a cylinder and parallel to a given straight line. (A plane containing the given line and a line par-

allel to the elements of the cylinder will be parallel to the required tangent plane.)

4. Given the top and front views of a point and a right circular cylinder which has its axis parallel to the ground line. Represent a plane which contains the point and is tangent to the cylinder.

5. Represent a plane which makes 45° with H and is tangent to a right circular cylinder. The axis of the cylinder is parallel to H and makes 30° with V.

CONES

111. A cone is the path of a straight line which moves so as to touch a curve and pass through a fixed point. If the directrix is a plane curve, the fixed point does not lie in the plane of the curve.

The curve is the **directrix,** the fixed point the **vertex,** and the moving line the **rectilinear generatrix** of the cone.

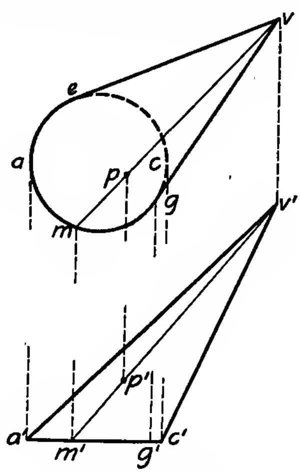


FIG. 66.—*Cone.*

The directrix can be a closed curve, as a circle or an ellipse, or it can be a curve which is not closed, as a hyperbola or a helix.

The generatrix being indefinite in length, will generate a surface of two parts which are on opposite sides of the vertex and which are called the **upper and lower sheets.**

The intersection of the cone with a horizontal plane is usually taken as the **base,** but any plane section of the cone may be considered the base. Cones are distinguished by the names of their bases; as circular cones, elliptical cones, etc.

A right cone or **a cone of revolution** is one in which all the rectilinear elements make the same angle with a straight line through the vertex called the **axis** of the cone. In the oblique cone, the line which joins the vertex with the center of the curve taken for the base will be considered the axis.

Any plane passing through the vertex and intersecting the cone will cut rectilinear elements from the surface.

112. To represent the surface. A cone is usually represented by the views of its base, vertex, and extreme elements.

In Fig. 66, AECG is the base and V the vertex of the cone. The extreme elements in the top view are tangent to the base and pass through the top view of the vertex. The extreme elements in the front view arc drawn through the extreme points of the front view of the base and through the front view of the vertex. It will be noted that the elements tangent to the base in the top view are not the same as the extreme or limiting elements in the front view.

113. To represent a rectilinear element of the surface. Through any point M of the base, draw a straight line through the vertex V; this will be an element of the surface.

To represent a point of the surface. Locate a rectilinear element, as MV, and take any point P of this element.

114. To represent a plane which contains a given point and is tangent to a cone.

The tangent plane to a cone is tangent all along a rectilinear element.

The line of intersection of the tangent plane with the plane of the base is tangent to the curve of the base.

Let the cone be given as in Fig. 67, and let P be the given point.

Since the required plane must contain a rectilinear element of the cone, and therefore the vertex, a straight line joining the vertex with the given point P will lie in the tangent plane and pierce the plane of the base at M. A line through the point M and tangent to the base of the cone is a horizontal line of the required tangent plane. An element through the point of tangency O is the element of contact. This element, the horizontal line, and the line VP all lie in the required tangent plane.

There are as many solutions to the problem as there are tangents to the base of the cone from the point in which the auxiliary line pierces the plane of the base. The problem cannot be solved when the given point is inside the cone.

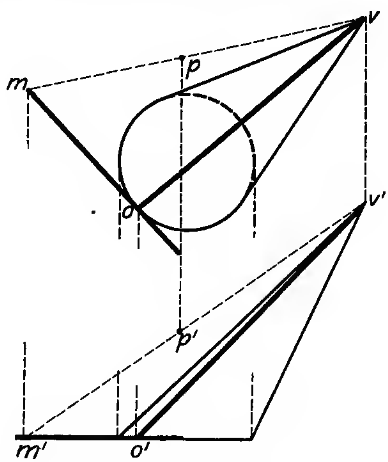


FIG. 67.—*Tangent plane to cone.*

115. Problems.

1. A right cone and a point are given by their top and front views. Represent a plane which contains the point and is tangent to the cone.

2. A right cone with its base in a profile plane and a point are given by their top and front views. Represent a plane which contains the point and is tangent to the cone.

3. A right cone with its axis parallel to H and oblique to V and a point are given by their top and front views. Rspresent a plane which contains the point and is tangent to the cone.

4. Represent a plane which is tangent to a given oblique cone at a point on the surface.

5. Represent a plane which is tangent to a given cone and parallel to a given straight line. (A line through the vertex of the cone and parallel to the given line will lie in the required tangent plane.)

CONVOLUTES

116. A convolute is the path of a straight line moving along and remaining tangent to a curve of double curvature such as the helix. The surface, Fig. 68, is represented by drawing the

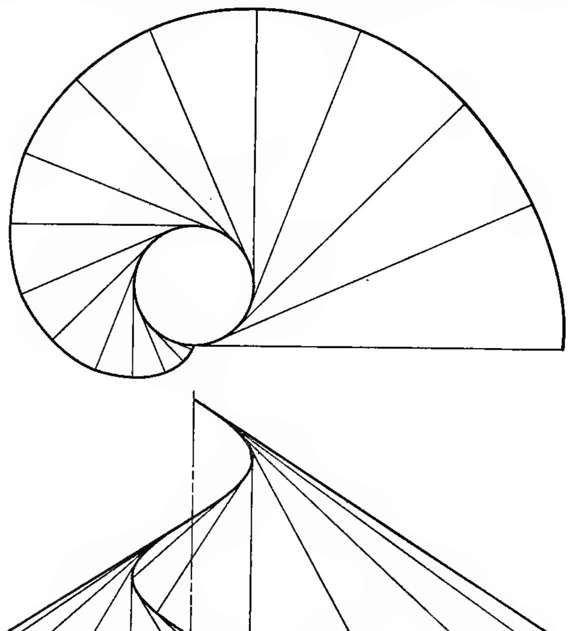


FIG 68.—*Convolute.*

views of the curvilinear directrix, an occasional element, and the intersection of the surface with a horizontal plane. If the curvilinear directrix is a helix with axis perpendicular to the horizontal plane, the base of the surface will be the involute of the circle

which represents the top view of the helix. When the directrix is a helix, the surface is also called a developable helicoid.

This surface is of very little practical importance, so no further discussion of it will be given. For further information on the convolute, see Salmon's Geometry of Three Dimensions, Art. 349.

WARPED SURFACES

117. There is a great variety of warped surfaces, differing from each other in their mode of generation and properties. Only a few of the more common ones will be discussed.

There are several kinds of warped surfaces formed by moving a straight line to touch two other lines, straight or curved, and keeping it parallel to a given plane, called a **plane directer**.

Warped surfaces are usually represented by drawing the views of one or more curves of the surface and a few of the rectilinear elements. The views of the directrices are often given.

HYPERBOLOIDS OF REVOLUTION OF ONE SHEET

118. A hyperboloid of revolution of one sheet is the path of a straight line revolving about another straight line as an axis, the generatrix and axis not being in the same plane.

From the nature of the motion it is evident that no two successive positions of the generatrix can be brought into one plane without distortion, therefore the surface is warped (Art. 98).

This is the only warped surface of revolution.

119. To represent the surface. Let the surface be formed by the rotation of line CE about the axis AB, Fig. 69. Each point of CE as P will move in the circumference of a circle with center on AB and plane perpendicular to AB. The top view of this path is the circle p p_1 p_2 and its front view is the straight line p_1' p'_2. The points P_1 and P_2 are in the meridian section MN parallel with V and are therefore on the curve representing the surface in the front view. By selecting other points on the generating line CE and revolving them into the plane MN about AB as an axis, other points on the curve representing the surface in the front view can be found. The hyperbola drawn through these points represents the surface in the front view. If this hyperbola be revolved about AB as an axis it will generate the surface of the hyperboloid. In the top view, a perpendicular to *ce* from the point *ab* locates the point *o*. The point O in space is the point of CE which is nearest the axis AB, and will therefore generate the smallest circle of the surface. This is called the **circle of the gorge.** The circle formed by any other point of the line CE as C may be considered the **base of the surface.** The top view of the base and the circle of the gorge is the top view of the surface.

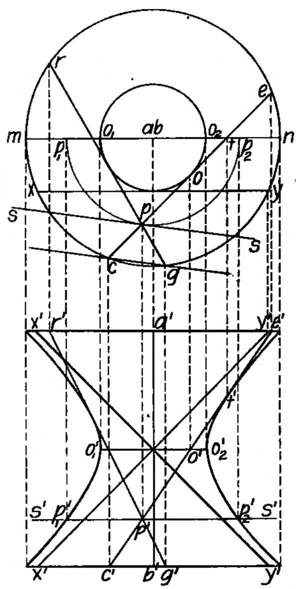


FIG. 69.—*Hyperboloid of revolution of one sheet*

The generating line CE pierces the meridian plane MN at the point T. $c'e'$ is tangent to the hyperbola representing the surface in the front view at t'. In a similar manner the front views of all rectilinear elements of the surface will be tangent to the

hyperbola. The element XY is parallel to the plane MN, and therefore its front view $x'y'$ is an asymptote of the hyperbola. There are two elements of the surface having xy for a top view, but different front views both lettered $x'y'$. If both of these lines are rotated about AB as an axis they will each generate the same surface. It is evident from this that **there are two rectilinear elements through every point of the surface.**

120. To represent an element of the surface. Draw the top view of any rectilinear element as ce tangent to the top view of the circle of the gorge. The front view of C is on the lower base at c' and that of E on the upper base at e'. Then $c'e'$ is the front view of the rectilinear element. The intersection o' of $c'e'$ with the front view of the circle of the gorge should be directly below the point of tangency o. To represent a circle of the surface, draw its top view p_1pp_2 and from this determine its front view $p_1'p_2'$. There is another circle of the surface, the distance above the circle of the gorge that $p_1'p_2'$ is below it, which has the same top view p_1pp_2. The front view of the circle could have been drawn first and then the top view found.

To represent a point of the surface. Select either the top or front view of the point and through it draw the corresponding view of either the rectilinear or circular element. Find the other view of the element, and then the corresponding view of the point on it.

121. To represent a plane tangent to the hyperboloid of revolution of one sheet at a point of the surface.

Let the surface be given as in Fig. 69, and let P be the point of tangency represented as in the preceding article.

Since there are two rectilinear elements, PE and PR, of the surface passing through the point P, they will represent the tangent plane to the surface. One of these rectilinear elements together with the tangent SS to the circular element through P would also represent the tangent plane. It is evident that the plane PER cuts the surface of the hyperboloid and is tangent at no other point but P. Each line on the surface through P, either

straight or curved, will have its tangent at this point lying in the plane PER. The plane is therefore tangent to the surface at this point. It is a characteristic of a tangent plane to a warped surface that **it contains the rectilinear element of the surface through the point of tangency, but the plane is tangent to the surface at but one point of the element.**

HELICOIDS

122. A helicoid is a surface formed by a straight line moving about an axis in such a manner that each point of the moving line traces a helix; all the helices must have the same axis and the same pitch. The generating line may or may not intersect the axis of the surface.

If the generatrix CE, Fig. 70, forms an acute angle with the axis AB, an **oblique helicoid** is formed, as in the triangular screw thread. If CE is perpendicular to the axis, a **right helicoid** is formed, as in the square screw thread.

From the nature of the motion it is evident that in general no two successive positions of the generatrix can be brought into one plane without distortion, therefore the surface is warped. In the particular case when the generatrix is tangent to a helix traced by one of its points, the surface generated is **a developable helicoid** or **convolute** which is a single curved surface.

123. To represent the surface.

Let AB, Fig. 70, be the axis, CE the initial position of the generating line, and the distance $a'b'$ the pitch of all the helices traced by points of the generatrix CE. $a'q'o'y'b'$ is the front view of the helix traced by the point of the generatrix O which is nearest the axis. The views of the helix are drawn by the method of Art. 96. In a similar manner the helices $r'f'c'$ --- and $g'e'z'$ --- traced respectively by the points C and E are represented. From the nature of the motion of the generatrix CE, it is evident that its top view in every position is tangent to the circle with radius *ao*. Therefore to represent a rectilinear element of the surface, draw its top view *fg* tangent to the circle with radius *ao*. The front view of F is at f' on the front view of the helix traced by the point C. Likewise, the front view of G is at g' on the helix traced by the point E. Then $f'g'$ is the front view of an element of the surface. The front views of other elements are drawn in a similar manner and then the curve $p'q'r'$ is drawn tangent to the front views of these elements. In a similar manner the curve $x'y'z'$ is determined. These curves show the contour of the helicoid in the front view.

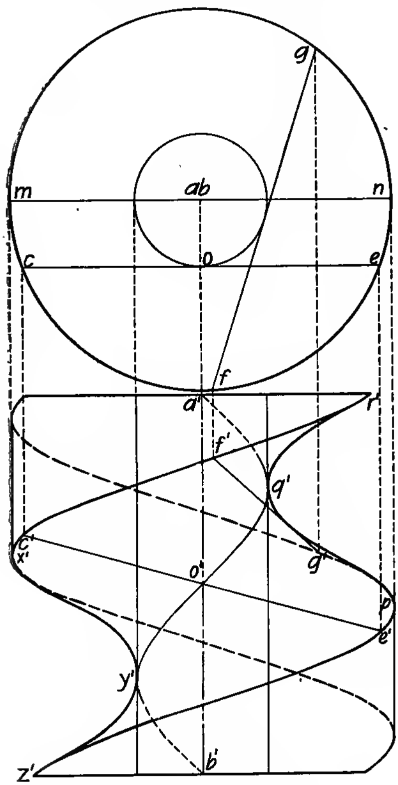


FIG. 70.—*The helicoid*

The portion of the helicoid usually represented is generated by a line of definite length as CE, and the top view of such a surface is the two concentric circles shown in Fig. 70. Sometimes the generating line is extended indefinitely in one direction from the point O, and the intersection of the surface with a horizontal plane, as the one through B, taken as the base. The base in this case is a spiral. Points of the spiral are found first in the front view, where the elements of the surface pierce the plane of the base, and are then projected to the top view where the spiral shows in its true size.

The helicoid can be varied in form by changing the distance the generatrix CE is from the axis AB, the angle CE makes with the axis, or the pitch of the helix $a'o'b'$. In the surface of the ordinary screw thread, the generatrix cuts the axis. Both surfaces of the screw thread may or may not be generated by one line or this line prolonged beyond the axis.

124. To represent a plane tangent to a helicoid at a point of the surface.

Let the surface be represented as in Fig. 70, and let G be the given point.

Since the surface is ruled, the straight line element FG will lie in the tangent plane. The helix traced by the point E passes through G, and therefore a tangent to the helix at G will lie in the tangent plane. These two lines determine the required plane. If the point G had not been on the helix traced by E, it would have been necessary to construct the helix through G and then draw a tangent to it.

The helicoid being a warped surface, the plane which is tangent to it at G will not be tangent at any other point of the element FG.

HYPERBOLIC PARABOLOIDS

125. A hyperbolic paraboloid is the path of a straight line moving along two straight lines not in the same plane, and remaining parallel to a given plane.

The moving line is called the **generatrix,** the two fixed straight lines the **directrices,** and the given plane the **plane directer.** Any position of the generatrix is called an **element of the surface.**

This surface has a second rectilinear generation in which any two rectilinear elements of the first generation may be taken as directrices and a plane which is parallel to the first directrices as a plane directer. It follows that **through any point of a hyperbolic paraboloid, two rectilinear elements can always be drawn.**

Any intersection of the surface by a plane, which is not a straight line intersection, is a hyperbola or a parabola, hence the name of the surface.

It is evident from the nature of the motion of the generating line that the surface is warped.

The pilot or "cow catcher" of an American locomotive is usually of the form of the hyperbolic paraboloid.

126. To represent the surface. In Fig. 71, AB and CD are the rectilinear directrices and V is taken for the plane directer. The plane directer must either be indicated or represented. Rectilinear elements are drawn parallel to the plane directer and cutting the directrices. In this figure the top views of the elements are first drawn and then their front views determined. A plane parallel with the plane directer will always cut the directrices in two points and a straight line joining these points will be an element of the surface. mn and $m'n'$ are the top and front views respectively of an element of the surface.

To represent a point of the surface, a rectilinear element, as MN, Fig. 71, is first drawn and then a point P of this element is taken.

127. To represent a plane which is tangent to the surface at a point of the surface.

Let O, Fig. 71, be the given point.

Analysis. Through the given point pass planes parallel respectively to the elements of each generation. These planes will

intersect the surface in the two rectilinear elements which determine the tangent plane at the given point.

Construction. The plane through O parallel to the elements of the first generation will cut the element VW from the surface. Through O draw OG parallel to CD, and OU parallel to AB. These lines will determine a plane parallel to the elements

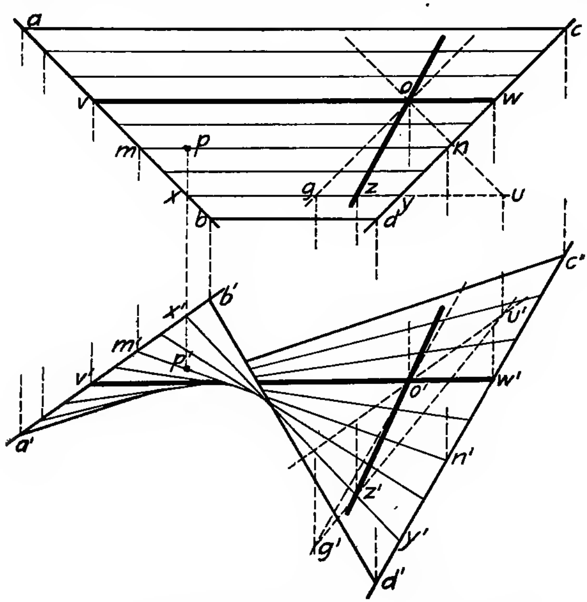


FIG. 71.—*Hyperbolic paraboloid and tangent plane.*

of the second generation. This plane will intersect the element XY in Z. The line joining Z with O will be an element of the second generation. OW and OZ will therefore determine the required tangent plane to the surface at the point O.

CONOIDS

128. A conoid is the path of a straight line moving along two other lines, one straight and the other curved, and remaining parallel to a given plane.

When the generatrix of a right helicoid touches the axis, the helicoid is also a conoid. In this form the coinoid has many practical applications.

To represent the surface. In Fig. 72, MN is the straight line and ABCD the curved line directrix. V is the plane directer. In this case, the top view of MN and the circle ABCD represent the top view of the surface. The front view of the surface is the triangle $m'a'c'$.

An element of the surface is represented by selecting some point of the curved directrix as E and drawing a straight line through this point parallel to the plane directer until it cuts the directrix MN at M. Then *em* is the top and $e'm'$ the front view of an element of the surface.

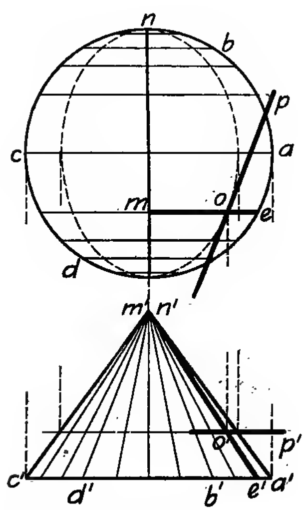


FIG. 72.—*Conoid and tangent plane.*

To represent a point of the surface, an element is first represented and then the point placed on the element. *o* and o' are the views of a point represented in this manner.

129. To represent a plane which is tangent to the conoid at a given point on the surface. Let O, Fig. 72, be the given point. The tangent plane must contain the rectilinear element EM through the point of contact. It must also contain the line OP tangent to the elliptical section made by a horizontal plane through the point. EM and OP will therefore represent the required tangent plane.

CYLINDROIDS

130. A cylindroid is the path of a straight line moving along two curves, and remaining parallel to a given plane.

The fender of some automobiles is a cylindroid or a surface similar in form to one. The surface is also used in architecture for joining the arched ceilings in two parallel corridors of different levels.

To represent the surface. In Fig. 73, ABC and DEG are the curved directrices and H is the plane directer. The surface is represented in the front view by the curved directrices and a series of horizontal lines. The top view of the surface consists of the top views of these straight line elements and the curved directrices. If enough straight line elements are drawn in the top view, the contour of the surface will show as a curved line touching all the top views of these elements.

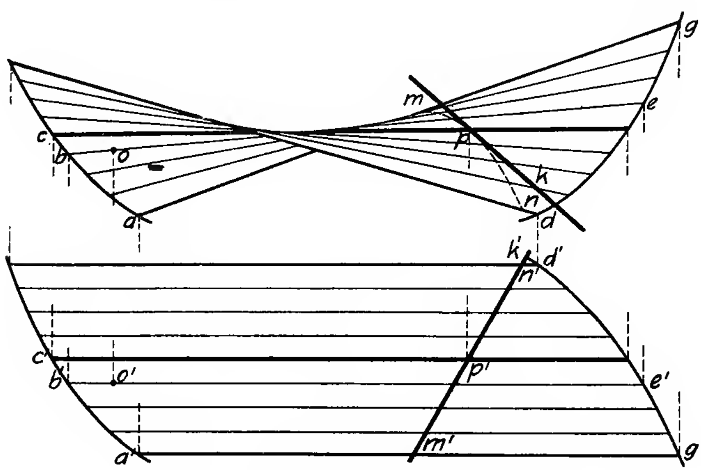


FIG. 73.—*Cylindroid and tangent plane.*

An element of the surface is represented by selecting some point B of one directrix and drawing a straight line parallel to the plane directer H until it cuts the other directrix at E.

Then $b'e'$ is the front and be the top view of a rectilinear element of the surface.

A point of the surface is represented by first representing an element of the surface as BE and then taking a point O of this element. o and o' are the views of a point on the surface.

131. To represent a plane which is tangent to the surface at a given point of the surface. Let P, Fig. 73, be the given point. The tangent plane must contain the rectilinear element CP through this point and the tangent to a plane section of the surface at P. NPM is a section of the surface by a plane perpendicular to V. pk is the top and $p'k'$ the front view of the tangent to this curve at P. Then CP and PK represent the tangent plane to the surface at the point P.

DOUBLE CURVED SURFACES

132. A double curved surface is a surface which can only be generated by the motion of a curved line. Such a surface has no straight line elements. The surface may be double convex as a sphere or ellipsoid, concave outward as the curved surface of a pulley, or concavo-convex as the annular torus or anchor ring.

The more common form of a double curved surface is that of revolution; that is a surface which has no straight line elements and can be formed by the rotation of some curve about an axis. The general form of the double curved surface, however, is not one of revolution. For example, the general form of the ellipsoid is a surface in which the three principal axes through the center are all of different length, while in the ellipsoid of revolution two of these axes are of equal length.

Since the general form of the surface is not common in practical work, **only double curved surfaces of revolution will be considered in this text.**

133. Some of the simple double curved surfaces of revolution are the following:

A sphere is the path of a circle revolving about its diameter as an axis.

An annular torus or anchor ring is the path of a circle revolving about a straight line which lies in the plane of the circle but does not cut the circumference.

An ellipsoid of revolution is the path of an ellipse revolving about either axis. When the ellipse is revolved about its major axis, a **prolate ellipsoid** is generated; when about its minor axis, an **oblate ellipsoid**. These surfaces are sometimes called **spheroids.**

A paraboloid of revolution is the path of a parabola revolving about its axis. This surface is extensively used for light reflectors.

A hyperboloid of revolution of two sheets is the path of a hyperbola revolving about the axis which passes through the foci.

134. To represent the surface. For convenience of representation, surfaces of revolution are usually taken with the axis of rotation perpendicular to one of the principal planes of projection, although the surface may be placed in any other position. The view of the surface taken by looking in the direction of the axis of rotation is one or more circles. If the surface is limited the largest circle is shown, but if the surface is indefinite in extent its intersection with some plane at right angles to the axis is shown. The view of the surface takeu by looking at right angles to the axis is the meridian line, the plane of which is parallel to the picture plane. For example, a prolate ellipsoid with its axis of rotation perpendicular to H, will have for its top view a circle with diameter equal to the minor axis of the generating ellipse and for its front view an ellipse identical to the generating ellipse, and having its major axis perpendicular to G. L.

Since there are no straight lines on a double curved surface of revolution, a circle of the surface with its plane perpendicular to the axis may be considered the element. A view of the

element looking in the direction of the axis is a circle, and the view looking at right angles to the axis is a straight line equal in length to the diameter of the circle, and having its extremities in the contour of the surface.

If one view of a point on a surface of revolution is given, the other view can be found as follows: Through the given view of the point draw the corresponding view of an element of the surface. Then find the other view of the element and place the required view of the point on it. It must not be overlooked that if the top view shows the true size of a circular element, the front will show it as a straight line parallel to G. L. and vice versa.

135. Problems.

1. Represent a point on the surface of a sphere.
2. Represent a point on the surface of a prolate ellipsoid.
3. Represent a point on the surface of an oblate ellipsoid.
4. Represent a point on the surface of a paraboloid of revolution.
5. Represent a point on the surface of a torus.

136. Tangent planes to double curved surfaces of revolution. The tangent plane to a surface at a point on the surface is determined by two straight lines tangent at this point to two lines of the surface. The simplest curves passing through the point on a surface of revolution are usually the meridian line and the circle which is cut from the surface by a plane perpendicular to the axis of the surface. The tangent plane, therefore, is determined by two straight lines, one tangent to the circular section at the given point and the other tangent to the meridian section at that point.

137. To represent a plane which is tangent to an ellipsoid of revolution at a point of the surface.

Let the surface be given as in Fig. 74 and let P be the given point represented by the method of Art. 134.

Analysis. A tangent to the meridian curve at the given point and a tangent to the circle of the surface at this point will represent the tangent plane (Art. 136).

Construction. prx is the top and $r'x'$ the front view of the circle of the surface through the given point P. pm is the top and $p'm'$ the front view of the tangent to the circle at the point P. ap is the top view of the meridian section through P. If

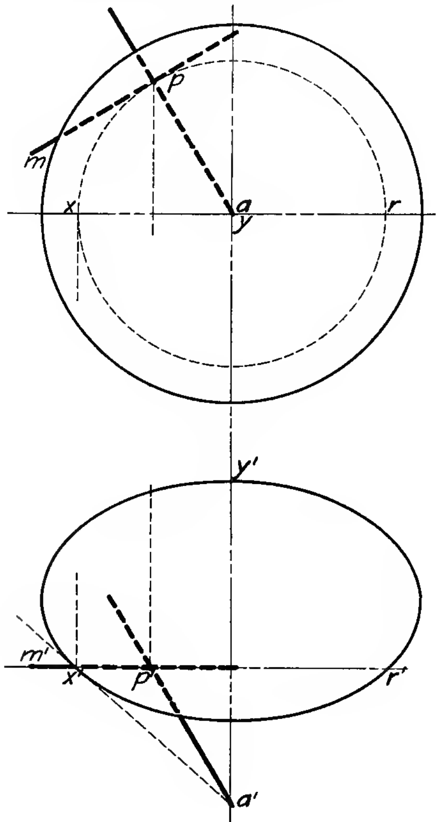


FIG. 74.—*Ellipsoid of revolution and tangent plane.*

the meridian curve through P is revolved about the axis of the surface until the plane of the curve is parallel to V, it will have the ellipse $r'x'y'$ for its front view, P moving to X. At x' draw

$x'a'$ tangent to the ellipse $x'r'y'$. This cuts the front view of the axis of the surface at a'. Then $x'a'$ is the front view of the revolved position of the tangent to the meridian section at P. In the counter revolution of the meridian plane, X moves to P and the point A, in the axis, remains fixed. Then ap is the top and $a'p'$ the front view of the tangent to the meridian curve at P. PM and PA represent the required plane which is tangent to the surface at the point P.

There are any number of planes tangent to a double curved surface of revolution from a point outside the surface. One or more of these planes can be determined by the general method employed in the previous paragraph.

138. Problems.

1. Represent a plane which is tangent to a sphere, (a) at a point of the surface; (b) and which contains a point outside the surface.
2. Represent a plane which is tangent to a prolate ellipsoid, (a) at a point of the surface; (b) and which contains a point outside the surface.
3. Represent a plane which is tangent to a paraboloid of revolution, (a) at a point of the surface; (b) and which contains a point outside the surface.
4. Represent a plane which is tangent to a torus, (a) at a point of the surface: (b) and which contains a point outside the surface.

139. To represent a plane which contains a given straight line and is tangent to a sphere.

Let AB, Fig. 75, be the given straight line and let C be the center of the given sphere.

Analysis. Assume that the required plane is drawn through the given line and tangent to the sphere. Now if an auxiliary plane is passed through the center of the sphere and perpendicular to the line AB, it will cut a point from AB, a great circle from the sphere and a line from the tangent plane which will pass through the point on AB and be tangent to the great circle cut from the sphere. Therefore, to make the construction for the tangent plane, pass a plane through the center of the sphere and perpendicular to the line AB. From the point in which this plane cuts AB, draw a tangent to the great circle cut from the sphere. This tangent and the line AB determine the re-

quired tangent plane.

Construction. SS and FF represent a plane which contains the center of the sphere and is perpendicular to the line AB. P is the point in which the line AB pierces this plane. When the plane of SS and FF, which contains a great circle of the sphere and the point P, is revolved about SS as an axis until it is parallel to H, the point P moves to P_2 and the great circle has the same top view as the sphere. p_2o_2 is the top view of the revolved position of a tangent from P to the great circle. By counter revolution the views *po* and $p'o'$ of the required tangent are found. Then PO and AB represent the required tangent plane.

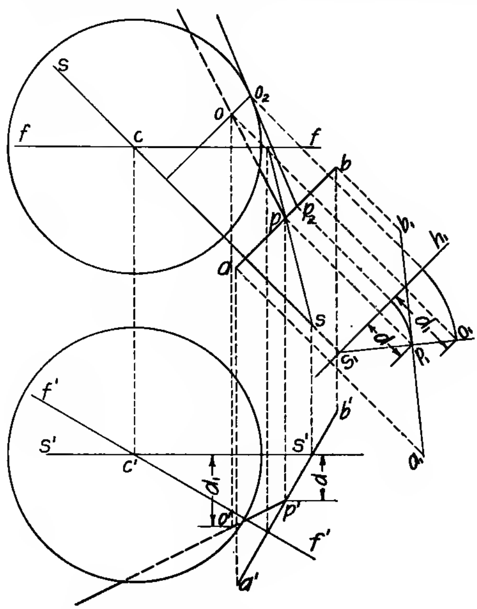


FIG. 75.—*Plane containing AB and tangent to sphere.*

There is another plane containing AB and tangent to the sphere. This plane is represented by AB and the other tangent from P to the great circle cut from the sphere by the plane of SS and FF.

140. Problems.

1. Represent a plane which contains a line parallel to V and is tangent to a given sphere.
2. Represent a plane which contains a line parallel to H and is tangent to a given sphere.
3. Represent a plane which contains a line parallel to G. L. and is tangent to a given sphere.
4. Represent a plane which contains a line of profile and is tangent to a given sphere.

GENERAL PROBLEMS

Solve the following problems in a 10″ x 14″ rectangle with the base line through the center and parallel to the shorter side, unless statement to the contrary is made.

1. P($\frac{1}{4}$, 3, —1) is the vertex of a cone whose base is a 2″ circle in a horizontal plane with center at O(—1$\frac{1}{4}$, $\frac{3}{4}$, —4$\frac{1}{2}$). Represent a plane which makes 60° with H, and is tangent to the cone. (Use a 7″ x 10″ rectangle).
2. A cone whose base is a 2″ circle in V, has A(—$\frac{1}{4}$, $\frac{3}{4}$, —3) B(1$\frac{1}{4}$, 4$\frac{1}{4}$ —$\frac{3}{4}$) for an axis. Find a plane which is tangent to the cone and makes 60° with V. (Use a 7″ x 10″ retangle).
3. The base of a cylinder is a 2$\frac{1}{2}$″ circle in H. L(1$\frac{1}{2}$, $\frac{1}{4}$, —4$\frac{1}{2}$) K(—$\frac{3}{4}$, 3, —1$\frac{1}{4}$) is the axis. Represent a plane which is tangent to the cylinder and makes 60° with H. (Use a 7″x10″ rectangle).
4. A(—1$\frac{1}{2}$, 1, —$\frac{3}{4}$) B($\frac{3}{4}$, 4$\frac{1}{4}$, —3$\frac{1}{2}$) is the axis of a cylinder whose base is a 2$\frac{1}{2}$″ circle in V. Find a plane which is tangent to the cylinder, and makes 60° with V. (Use a 7″ x 10″ rectangle).
5. Represent a plane which is tangent to a 3″ sphere with center at O(—$\frac{1}{2}$, 4$\frac{3}{4}$, —2$\frac{1}{2}$) and is parallel to the plane A(—3, 4$\frac{3}{4}$, —5$\frac{1}{4}$) B(0, 1$\frac{3}{4}$, —5$\frac{1}{4}$) C(—3$\frac{1}{2}$, 1$\frac{3}{4}$, —2$\frac{1}{2}$).
6. Inscribe a sphere in a given tetrahedron.
7. A(—3, 2, —5$\frac{1}{2}$) B(—$\frac{1}{2}$, 3$\frac{1}{4}$, —2$\frac{3}{4}$) C(1, 1, —3$\frac{3}{4}$) is a plane which is tangent to a right circular cylinder whose axis is E(—$\frac{1}{2}$, 2$\frac{3}{4}$, —5$\frac{1}{4}$) G(1$\frac{3}{4}$, 4, —2$\frac{3}{4}$). Draw the top and front views of the element of contact.
8. Through a given point, pass a plane tangent to a cylinder of revolution, having given the axis and radius of the cylinder, without finding the views of the cylinder.

9. Having given the radius and the axis of a cylinder of revolution, pass a plane tangent to the cylinder and parallel to a given straight line, without finding the views of the cylinder.

10. Find a common normal to two cylinders of revolution.

11. To two given cylinders pass tangent planes which are parallel to each other.

12. To a given cone and cylinder, pass tangent planes which are parallel to each other.

13. $A(-3, 1\frac{1}{2}\ -5\frac{3}{4})$ $B(-\frac{1}{2}, 4\frac{1}{4}, -2)$ and $C(2\frac{1}{2}, 2\frac{1}{4}, -5\frac{3}{4})$ $D(3\frac{1}{4}, 5\frac{1}{2}, -2)$ are the axes of two cones whose bases are $2\frac{1}{2}''$ and $3''$ circles, respectively, which lie in a horizontal plane. To the given cones pass tangent planes which will be parallel to each other.

14. Through a given point, pass a plane tangent to two given spheres.

15. Pass a plane tangent to two given spheres and parallel to a given straight line.

16. Pass a plane tangent to three given spheres.

17. Through a given point, pass a plane which is equidistant from three given spheres.

18. Through a given straight line pass a plane which is equidistant from two given spheres.

19. Pass a plane which is equidistant from four given spheres.

20. Find a common tangent plane to a sphere and a cylinder of revolution.

21. Find a common tangent plane to a sphere and a cone of revolution.

CHAPTER IV

PLANE SECTIONS AND DEVELOPMENTS OF CURVED SURFACES

141. Plane section. The line of intersection of an oblique plane with a curved surface which is ruled can be drawn by finding the points in which the rectilinear elements of the surface pierce the given oblique plane. The line joining in proper order these piercing points is the required line of intersection of the surface with the plane. These piercing points can be found by the usual method of finding where a straight line pierces a plane, or by the use of an auxiliary view. The auxiliary view should be taken from such a direction that the cutting plane will appear as a straight line. The rectilinear elements should be taken near enough together to give the line of intersection as accurately as desired.

The intersection of an oblique plane with a double curved surface is found by using a system of auxiliary planes. Each of the auxiliary planes will cut a straight line from the oblique plane and a curved line from the double curved surface and these lines will intersect in points of the required line of intersection. The auxiliary planes should be so taken that they will cut simple curves from the double curved surface and these curves should be in simple positions with reference to the planes of projection.

Development. By the development of a curved surface is meant a plane area of such form and size that it can be rolled or folded to again form the original surface. The development is obtained by rolling the surface upon a tangent plane until each part of the surface comes in contact with the plane. The part of the plane thus covered is the development of the surface.

142. To find the intersection of a right circular cylinder with a given oblique plane and to develop the surface of the cylinder.

Let the cylinder be given as in Fig. 76, and let SS and FF represent the oblique plane.

Analysis. Find the points in which the rectilinear elements of the cylinder intersect the given oblique plane. A curved line joining these points in the proper order is the required line of intersection.

Construction. A view taken by looking in the direction of the horizontal line SS, shows the cylinder as a rectangle and the oblique plane as the line s_1f_1. The distance at which the line

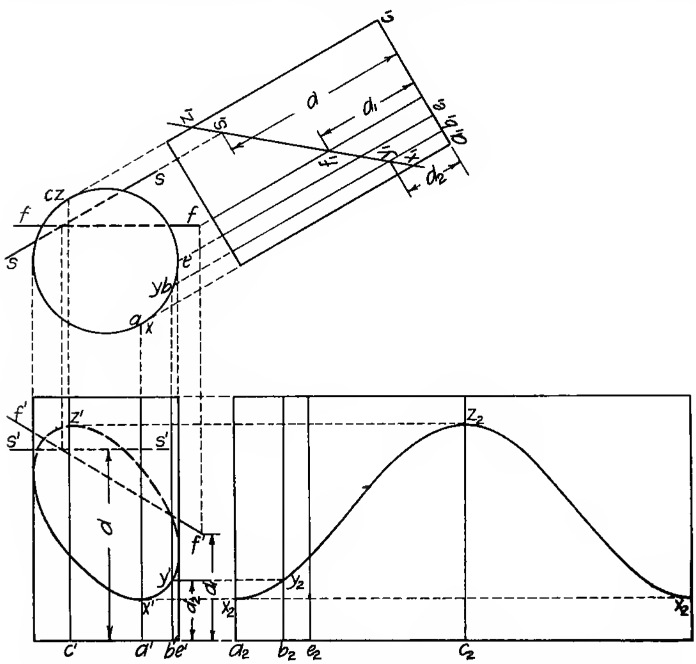


FIG. 76.—*Plane section and development of right circular cylinder.*

a_1c_1, representing the base of the cylinder, is placed from the top view is arbitrary. The distances d and d_1 are taken from the front view and used to locate the line s_1f_1 which represents the oblique plane in the auxiliary view. The front and auxiliary

views of elements of the cylinder are drawn from points in the base as A, B, C, etc., as many elements being drawn as are necessary to determine the curve of intersection to the desired accuracy. The points x_1, y_1, z_1, etc., where the auxiliary views of these elements intersect the line s_1f_1 are the auxiliary views of the points in which the elements pierce the cutting plane SS, FF. By taking the distances of the points x_1, y_1, z_1, etc., from the line a_1c_1 and setting them off on the proper elements from the base $a'c'$, the front views x', y', z', etc., of the points are located. Joining these points in the proper order gives the ellipse which is the front view of the intersection of the cylinder with the plane. The top view of the intersection is the circle a, b, c.

To develop the surface of the cylinder showing the curve of intersection with the oblique plane. If the cylinder be rolled on a tangent plane until each element has come into the plane, the base will develop into the straight line a_2c_2. The distance a_2b_2=arc ab, b_2e_2=arc be, etc. In the development, the elements will be perpendicular to a_2a_2, since they are perpendicular to the plane of the base. To get the line of intersection on the development, lay off $a_2x_2=a'x'$, $b_2y_2=b'y'$, etc. A smooth curve passing through the points x_2, y_2, z_2, x_2, is the development of the line of intersection.

143. To draw a straight line tangent to the curve of intersection of a plane with a curved surface at any point on the curve. The curve is evidently a plane curve and therefore a tangent to it at any point must lie in the plane which cuts the curve from the surface. Since the curve lies on the surface, a line which is tangent to it at any point must lie in a tangent plane to the surface at that point. (Art. 101). The required line is, therefore, the line of intersection of the cutting plane with the plane tangent to the surface at the point to which the tangent line is to be drawn.

This method is general and applies to the plane section of all surfaces. The construction for the problems is not as a rule,

difficult, since usually the tangent plane to the surface can easily be drawn.

In this chapter of the text, the lines tangent to the plane sections of the different surfaces will not be shown in the figures, but in every case they can, if desired, be drawn by the above method.

144. To find the intersection of an oblique cylinder with a given oblique plane and to develop the surface of the cylinder.

Let the cylinder be given as in Fig. 77, and let SS and FF represent the oblique plane.

Analysis. Take a view of the cylinder and plane by looking in the direction of the horizontal line SS. This view shows the cutting plane as a line. The points in which this line crosses the auxiliary view of the elements of the cylinder are the auxiliary views of points on the required curve of intersection. From the auxiliary view, the top and front views of the intersection can be derived.

Construction. In the auxiliary view, the base of the cylinder appears as the line a_1b_1 at right angles to ss, and the center of the base as the point c_1. The auxiliary view c_1o_1 of the axis is determined and then the extreme elements of the cylinder are drawn parallel to c_1o_1 through the points a_1 and b_1. The upper end of the cylinder is shown as broken. s_1f_1 is the auxiliary view of the oblique plane. The points where the auxiliary view of the elements cross the line s_1f_1 are the auxiliary views of the points where these elements pierce the cutting plane. The top views of these piercing points are found by first finding the top views of the elements and then locating the top views of the points on the top views of the corresponding elements. From the auxiliary and top views, the front views of the piercing points are found. A curved line joining these points in the proper order is the required line of intersection. The method for finding points on the curve of intersection would be the same, no

matter what direction the oblique plane SS FF is taken. It was drawn at right angles to the axis in order to be used in the development of the cylinder.

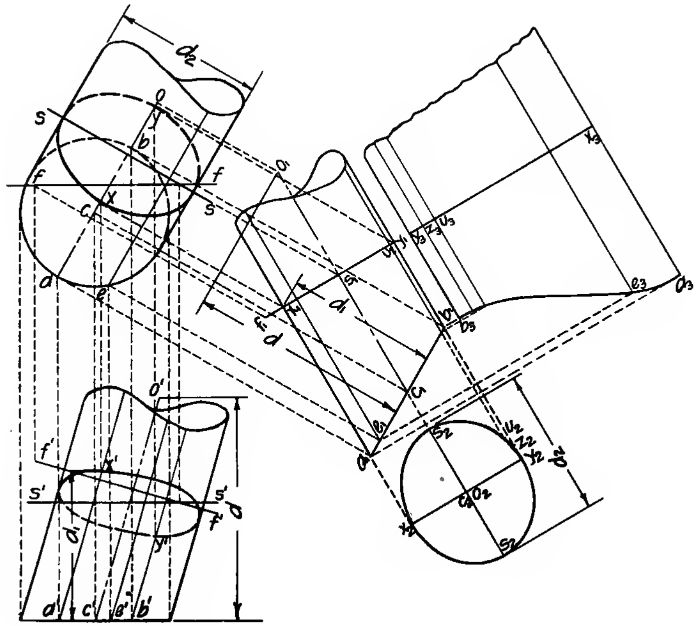


FIG. 77.—*Plane section and development of oblique cylinder.*

To develop the surface of the cylinder. If the cylinder be rolled on a tangent plane, the section which is at right angles to the axis will become a straight line on the development and the elements of the cylinder will appear in their true lengths and lie at right angles to this line. The true size of this right sec tion is found by a second auxiliary view, taken by looking in the direction of the axis, as shown in Fig. 77, or by rotating the cutting plane about SS as an axis until it becomes parallel to H. This would give the true size of the figure in the top view. From the true size of the right section the distances are

taken and laid off as follows along a straight line: arc $y_2z_2 = y_3z_3$, arc $z_2u_2 = z_3u_3$, etc. The true lengths of the elements of the cylinder are shown in the first auxiliary view. These lengths are taken and laid off on the development at right angles to y_3x_3 at the proper points along that line. In Fig. 77, the line y_3x_3 in the development was taken in the continuation of x_1y_1 in order that the true lengths of the elements could be projected across from the first auxiliary view to the development. In the development, the line y_3x_3 represents the right section and $b_3e_3a_3$ the curve of the base. Only half the cylinder is shown in the development.

145. To find the interesection of an oblique cone with a given oblique plane and to develop the surface of the cone.

Let the cone be given as in Fig. 78, and let SS and FF represent the oblique plane.

Analysis. Take a view of the cone and plane by looking in the direction of the horizontal line SS. This view shows the cutting plane as a line. The points in which this line crosses the auxiliary view of the elements of the cone are the auxiliary views of the points on the required curve of intersection. From the auxiliary view, the top and front views of the intersection can be derived.

Construction. $v_1a_1e_1$ is the auxiliary view of the cone and s_1f_1 that of the cutting plane. It is best to divide each half of the circumference of the base of the cone into the same number of equal parts, beginning at A, the foot of the shortest element. Then each element in the auxiliary view represents two elements of the cone, one visible and one invisible. By drawing the auxiliary views of the elements, the auxiliary views x_1, y_1, z_1, etc., of the points of intersection are found. From the auxiliary views, the top and then the front views are derived. A curve joining in proper order the points thus found, is the required line of intersection.

To develop the surface of the cone. It is first necessary to find the true lengths of the elements VA, VB, VC, etc. In Fig. 78, this is done by rotating the elements until they are parallel to the picture plane upon which the auxiliary view is taken.

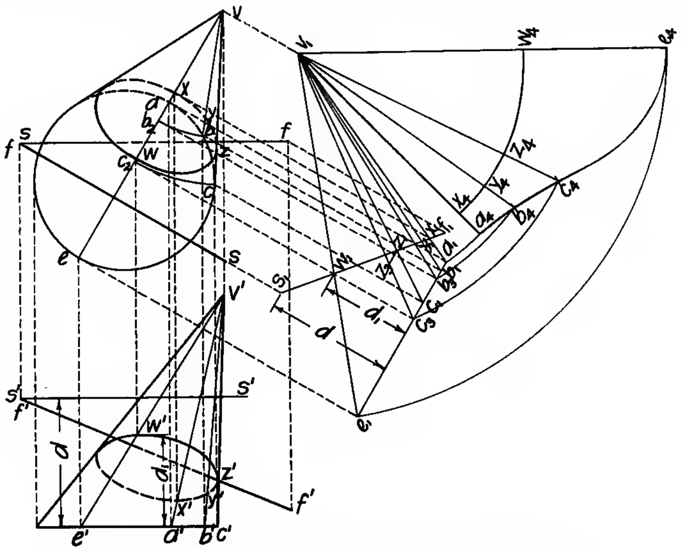


FIG. 78.—*Plane section and development of oblique cone.*

The true lengths are then found to be v_1a_1, v_1b_3, v_1c_3, etc. Through v_1 draw a line in any direction and lay off on it v_1a_4 equal to v_1a_1, the true length of the shortest element of the cone. With v_1 as center and v_1b_3 as radius strike an indefinite arc. With a_4 as center and radius equal to the rectified arc ab, strike an arc cutting the first arc at b_4. Then v_1b_4 is the element VB laid off on the development. By using v_1 as center and v_1c_3 as radius and b_4 as center and radius equal to the rectified arc bc, the point c_4 is located. By continuing the process the development as shown in Fig. 78 is found. The development for only half the cone is shown, the other half is similar. If the given cone had been right circular, the curve $a_4 \ldots e_4$ would have been the arc of a circle with center at v_1.

To lay the curve of intersection of the cone with the plane SS FF off on the devolpment, the true lengths of the elements from the vertex to the cutting plane are found and laid off on the corresponding elements of the development. This is easily accomplished by drawing lines through the points x_1, y_1, z_1, etc., parallel to a_1e_1, until they cut the true lengths of the corresponding elements. The construction is shown for the point Z, v_1z_3 being the true length from the vertex to the cutting plane. The distances thus found laid off on the development gives the curve $X_4 \ldots W_4$.

146. To find the intersection of a warped surface with a given oblique plane.

Analysis. Since a warped surface is ruled, it is only necessary to find where the rectilinear elements of the surface pierce the plane. A line joining in proper order these points is the required line of intersection. The most convenient method of finding these piercing points is by means of an auxiliary view which shows the cutting plane as a line.

The construction is left as an exercise for the student.

A warped surface cannot be developed. For an approximate development of such a surface, see Art. 148.

147. To find the intersection of any surface of revolution with a given oblique plane.

Let the surface be given as in Fig. 79 and let AB and AC represent the oblique plane.

Analysis. Cut the surface and plane by a system of auxiliary planes perpendicular to the axis of revolution of the surface. The auxiliary planes will cut circles from the surface which will intersect the lines cut from the given plane in points of the required curve of intersection.

Construction. Since the method is the same for all surfaces of revolution, let the construction be made for the torus as given in Fig. 79. The axis of the surface is perpendicular to H. The auxiliary cutting planes will therefore be parallel to H, and will be represented in the front view by lines parallel to G. L. $h'h'$

represents one of these planes. It cuts the line SS from the plane ABC and two circles from the torus, one with a radius $q'r'$ and the other with a radius $q'u'$. The top view shows these circles cutting ss at x, y, z, and w, the top views of four points on the required curve of intersection. The front views of these points are on the line $h'h'$. By taking other planes parallel to H, enough points can be found to determine the curve of the intersection to the desired accuracy.

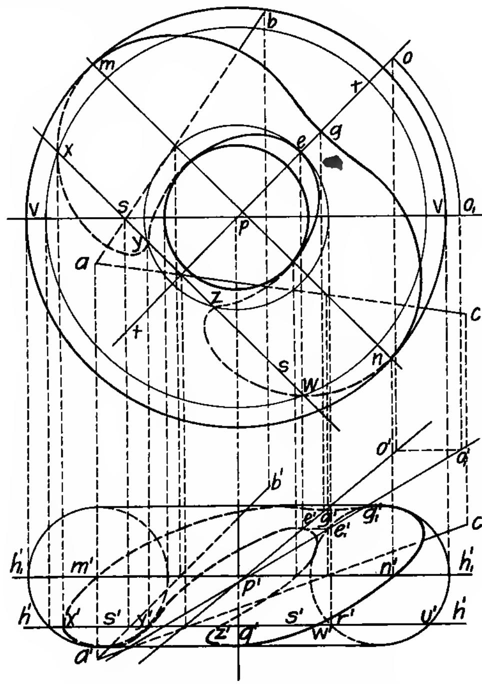


FIG. 79.—*Plane section of a torus.*

When the surface is symmetrical, as is the case with the torus, it is convenient to begin the construction with the plane which contains the center of the front view of the surface. This plane contains the largest and smallest circles of the torus as shown in the top view. If the remaining planes be passed in pairs at

equal distances above and below the center plane, each pair will cut circles having the same top view. This will avoid the necessity of drawing so many circles in the top view. The highest and lowest planes which touch the surface will contain but one circle each.

A plane which contains the axis of the surface and is parallel to V, cuts a line from the plane ABC which intersects the small circles of the front view of the torus at the points where the curve of intersection touches these circles. A plane T which contains the axis of the surface and is perpendicular to a horizontal line SS of the plane ABC will locate the points E and G on the curve of intersection. The points E and G are determined by rotating the section made by the plane T until it is parallel to V, locating the points e_1' and g_1', and then rotating back to the original position. For most surfaces of revolution, this section will locate the highest and lowest points on the curve of intersection of the oblique plane with the surface. Instead of rotating the plane T until it is parallel to V, the points E and G could have been located by taking an auxiliary view of the surface and plane ABC by looking in the direction of the line SS. This, however, would have involved more construction than the method by rotation.

148. Approximate development of warped surfaces. Since two consecutive straight line elements of a warped surface do not lie in one plane, the surface cannot be developed. In practice, however, approximate developments of these surfaces are made in the following manner.

Let the surface be a transitional piece joining a circle at one level with an ellipse at a different level, Fig. 80. The openings joined by the transitional piece need not be in parallel planes and they need not be of the form given in Fig. 80. If the surface is such that the straight line elements A1, B2, C3, etc., are neither parallel nor intersecting, it is warped and may be approximately developed by the following method.

Divide one quarter of the circle into any number of equal parts as six, Fig. 80. Divide the corresponding quarter of the ellipse into the same number of equal parts and join points on the ellipse with the points on the circle by the lines A1, B2, C3, etc.

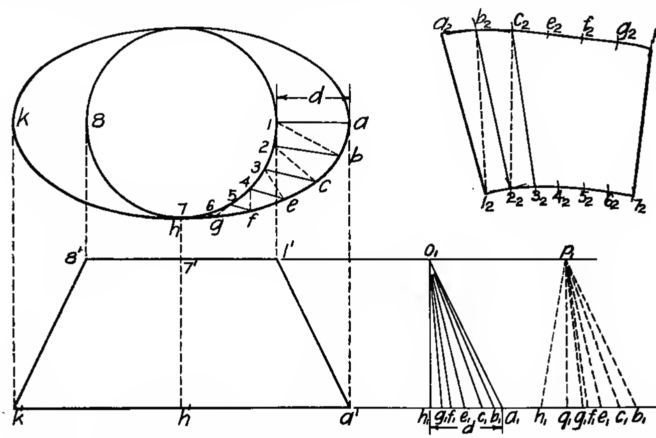


FIG. 80.—*Method for approximate development of warped surface.*

Draw the dashed diagonal lines B1, C2, E3, etc. Find the true lengths of both of these sets of lines. A convenient method for getting these lengths is shown in the sketches to the right of the front view. The lengths of the top views of the lines are laid off on the base line from the foot of the perpendiculars from o_1 and p_1. The development is obtained by drawing the line $a_2 1_2$ in any direction and laying it off equal to the true length of A1. With 1_2 as center and the true length of B1 as radius strike an arc. With a_2 as center and ab as radius strike an arc cutting the first arc at b_2. Then with b_2 as center and B2 as radius and 1_2 as center and *12* as radius strike arcs intersecting at the point 2_2. By continuing this process the development of the entire piece is found. Only one-quarter of the development is shown, but the remainder is found in a similar manner.

149. Approximate development of a sphere. Theoretically, double curved surfaces can not be developed, but approxi-

mate developments are made and used in practice. There are two methods used for developing a sphere.

Method I. **The gore method.** Let the sphere be given as shown in Fig. 81. Only half the surface is shown and only one quarter of the sphere is developed, but this will suffice to explain the method. Divide the surface into gores by the meridian planes A7, B7, C7, etc., sixteen is a good number

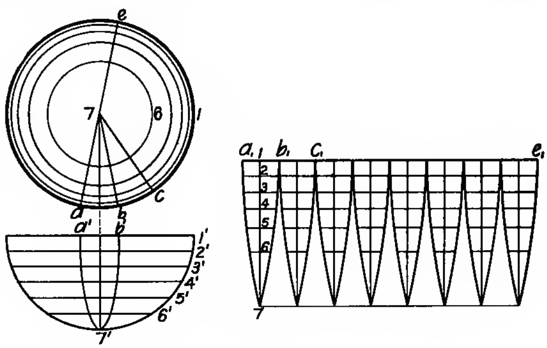


FIG. 81.—*Gore method of developing a sphere.*

although the more parts taken the more accurate the development should be. Then divide the surface by a series of horizontal circles 1, 2, 3, etc. In the development the great circle 1 becomes the straight line $a_1, b_1 \ldots e_1 \ldots$, the distance $a_1b_1 =$ arc ab, $b_1c_1 =$ arc bc, etc. The line *17* in the development is at right angles to a_1e_1, and is equal in length to one-quarter of a great circle, the distance *12* = arc *1'2'*, *23* = arc *2'3'*, etc. Center lines for each gore are drawn from the proper points along the line a_1e_1 and these center lines divided into parts similar to the first one by lines through points *2*, *3*, *4*, etc., parallel to a_1e_1. Step off from the center line of each gore along the lines *2*, *3*, *4*, etc., distances obtained from the corresponding circles in the top view, measured from the center line to the sides of the gore. Joining the points thus located gives the development shown in Fig. 81. The development of the complete sphere

would consist of twice as many gores as shown, the half of each gore above the line a_1e_1, being similar to the half shown below this line.

Method II. **The zone method.** Let the sphere be given as in Fig. 82. Half the sphere is shown in the top and front views and the same half in the development.

Divide the surface of the sphere into zones by the horizontal planes 1, 2, 3, etc. The greater the number of zones the greater should be the accuracy of the development. Each zone is then developed as if it were the frustrum of a right cone. The radius

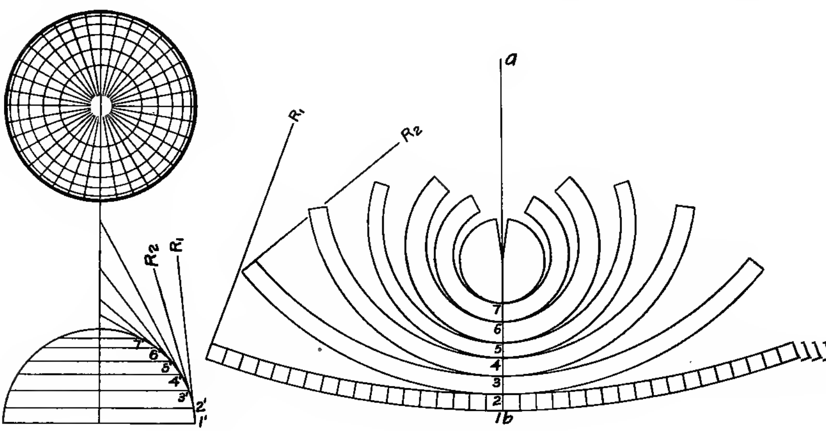


FIG. 82.—*Zone method for developing a sphere.*

of circle 1 in the development is the length of the line *1′2′* from the point *1′* to where *1′2′* cuts the the vertical axis of the sphere. Two circles pass through the point 2 in the development, one with a radius from *2′* to the point where the line *1′2′* cuts the axis of the sphere, and the other with a radius from the point *2′* to where *2′3′* cuts this axis. By continuing in this manner, all the circles shown in the development are drawn. It is best to take a center line *ab* and have the centers of all these circles on this line. Meridian planes are then passed through the axis of the sphere dividing the surface into any convenient num-

ber of parts as shown in the top view. With dividers, step off the distances on circle 1 between these planes in the top view upon circle 1 of the development, beginning with the center line *ab* and stepping half of them off on each side of this line. Each zone is stepped off in this way from the top view upon the development. In Fig. 82 division lines are shown for the first zone only.

The developed zones for the other half of the sphere would have centers on the line *ab* extended and would be curved in the opposite direction. The other half would be an exact duplicate of the half shown. Patterns cut from such a development should give an approximate sphere when bent into proper position.

PROBLEMS

The following problems can be conveniently solved in a 10″ x 14″ rectangle. In each case where co-ordinates are given the base line should be drawn through the center of the rectangle and parallel to the longer side.

In each of the following problems draw a straight line tangent to the curve of intersection at some point not on an extreme element of the surface.

1. The base of an oblique cylinder is a $2\frac{1}{2}$″ circle in H with center at C($-5\frac{1}{2}$, 1, $-4\frac{3}{4}$). The top view of the axis makes 60° with the ground line and the axis passes through O(-4, ?, $-\frac{3}{4}$). Develop the surface of the cylinder.

2. The base of a right circular cylinder which has its axis parallel to H and 60° with V is a $2\frac{1}{2}$″ circle with center at A($-5\frac{1}{4}$, $\frac{3}{4}$, -3). A horizontal line of a plane which makes 45° with H is perpendicular to the elements and cuts the axis $2\frac{1}{2}$″ from A. Develop the surface of the cylinder, showing the curve of intersection with the plane.

3. The base of a right cylinder is an ellipse in a horizontal plane. Find the intersection of this cylinder with a given oblique plane.

4. Cut a circle with diameter equal to the major axis of the ellipse from the cylinder in problem 3.

5. The base of an oblique cone is a $2\frac{1}{2}''$ circle in H with center at C$(-5\frac{1}{2}, 1, -4\frac{3}{4})$ and vertex at V$(-4, 3\frac{5}{8}, -\frac{3}{4})$, The top view of a horizontal line of a plane which makes 45° with H is perpendicular to the top view of the axis. Draw the top and front views of the line of intersection of the plane and cone and develop the surface of the cone.

6. A horizontal circle $3''$ in diameter is the base of an oblique cone. The axis of the cone is $5''$ long, makes 60° with H and is oblique to V. Find the intersection of the cone with a plane perpendicular to the axis and develop the surface of the cone, showing the curve of intersection on the development.

7. A right circular cone has its axis parallel to H and 60° with V. The radius of the base is $1\frac{1}{4}''$ and the altitude is $3\frac{1}{2}''$. Find the intersection of this cone with a plane which makes 45° with H and has its horizontal line at right angles to the axis of the cone. Also develop the surface of the cone.

8. From a right circular cone, with base in a horizontal plane, cut an ellipse and show the true size of the section.

9. From a right circular cone, with base in a horizontal plane, cut a parabola and show the true size of the section.

10. From a right circular cone with base in a horizontal plane, cut an hyperbola and show the true size of the section.

11. Is the shortest path around a cone, following the surface, starting at a point and returning to the same point, a plane curve?

12. Find the intersection of a sphere by an oblique plane.

13. Find the intersection of an oblate ellipsoid of revolution by an oblique plane.

14. Find the intersection of a prolate ellipsoid of revolution by a given oblique plane.

15. Find the intersection of a torus by a given oblique plane.

CHAPTER V

INTERSECTIONS OF CURVED SURFACES

150. To find the line of intersection of any two surfaces. In general, pass a system of planes through the surfaces. The planes will cut lines from each surface and the intersections of these lines will be points on the required line of intersection. The planes should be passed so that they will cut the simplest lines possible from the given surfaces.

Sometimes the intersection can be more easily found by using a system of spheres in place of the system of planes. For example, if the axes of two surfaces of revolution intersect, a system of spheres having their centers at the intersection of the axes and radii of different lengths, will cut circles from each of the original surfaces. The intersections of these circles are points on the required line of intersection.

When one of the surfaces is a cylinder, particularly a right circular cylinder, an auxiliary view taken by looking in the direction of the axis of the cylinder will be found a convenient method for getting the intersection.

151. To find the line of intersection of two right circular cylinders.

Let the cylinders be given as in Fig. 83.

Analysis. Pass a system of planes parallel to the axes of both cylinders. These planes will cut straight line elements from each cylinder which will intersect in points of the required curve of intersection.

Construction. The planes which are parallel to the axes of both cylinders appear as straight lines in the front and also in the auxiliary view. $t't'$ represents one such plane in the front view. This plane is located in the auxiliary view by taking the distance d which $t't'$ is above $m'n'$ in the front view and setting it above h_1h_1, thus locating t_1t_1 which is the auxiliary view of

the plane. The auxiliary view t_1t_1 might have been drawn first and then the front view $t't'$ determined. This plane cuts two elements from each of the cylinders, and these elements intersect locating the points X, Y, Z, and W, on the curve of intersection of the

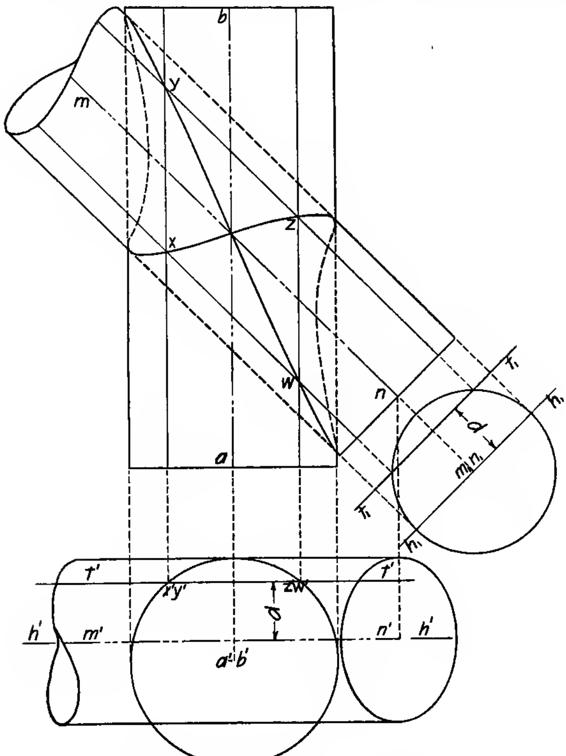


FIG. 83 —*Intersection of two cylinders.*

cylinders. By using other planes higher and lower than T, enough points can be located to determine the curve of intersection as shown in Fig. 83.

To determine which part of the curve of intersection is visible. A point on the curve of intersection is visible in the top view if it lies on an element of each cylinder which is visi-

ble in that view. In like manner, a point is visible in the front view when it lies on an element of each cylinder which is visible in that view. This will be found a convenient method for determining which part of the curve of intersection of any two surfaces is visible or invisible in any view.

152. To find the line of intersection of a right circular cone and a right circular cylinder.

Let the cone and cylinder be given as in Fig. 84.

Analysis. 1st method. Pass a system of planes through the vertex of the cone and parallel with the axis of the cylinder. These planes will cut straight line elements from each surface and these lines will intersect in points of the required curve of intersection.

2nd method. Pass a system of planes perpendicular to the axis of the cone. These planes will cut circles from the cone which will intersect the straight lines cut from the cylinder in points of th erequired curve of intersection.

Construction. 1st method. Take an auxiliary view by looking in the direction of the axis of the cylinder. In this view v_1e_1 represents a plane which passes through the vertex of the cone and is parallel to the axis of the cylinder. This plane cuts the elements MN and PQ from the cylinder and the elements VE and VG from the cone. These elements intersect in the points M, N, P, and Q of the required curve of intersection. In a similar manner other points can be located and the curve of intersection as shown in Fig. 84 determined.

2nd method. If the planes are passed according to the second method given in the analysis, they will appear as lines parallel to the base of the cone in the auxiliary view. h_1h_1 is the auxiliary view of one of these planes. It cuts the elements XY and WZ from the cylinder and a circle of radius o_1r_1 from the cone. The circle cuts the elements of the cylinder at the points X, Y, Z, and W of the required curve of intersection. Continuing in this manner, the curve of intersection as shown in Fig. 84 is determined.

When the surfaces are located as in Fig. 84, the second method gives a more accurate construction than the first.

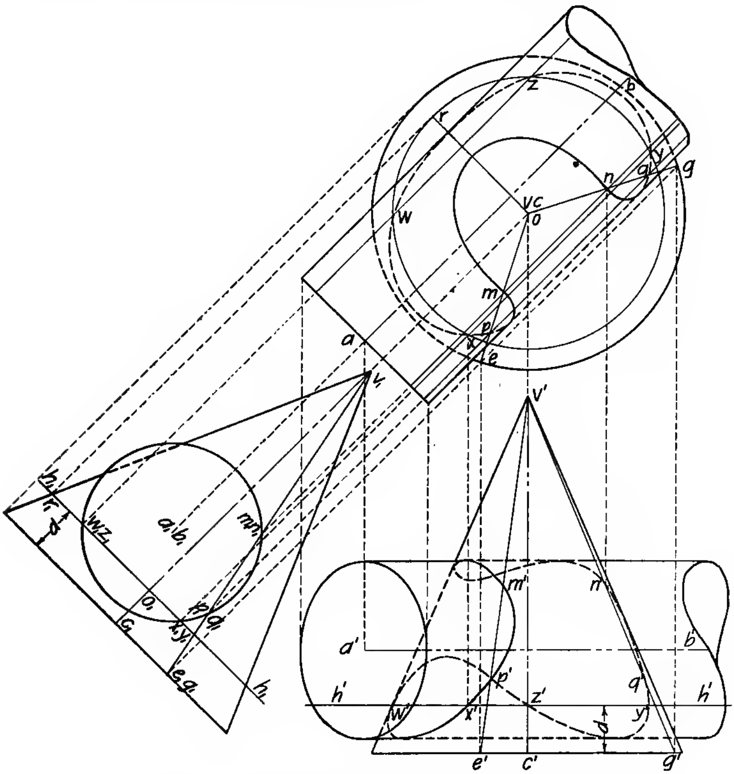


FIG. 84.—*Intersection of a cylinder and cone.*

153. To find the line of intersection of two oblique cylinders.

Let the cylinders be given as in Fig. 85.

Analysis. Pass a system of planes parallel to the axes of both cylinders. These planes will cut straight line elements from each cylinder which will intersect in points of the required curve of intersection.

Construction. Through any point A draw two lines, one parallel to the elements of one cylinder and the other parallel to the elements of the other cylinder. The plane of these lines cuts the plane of the bases of the cylinders in the line *tt*. Since the planes are parallel, each of them will cut a line from the plane of the bases which is parallel to *tt*. The plane T_3 which is tangent to one of the cylinders, cuts the element BB_1 from that cylinder and the elements CC_1 and DD_1 from the other cylinder. These elements intersect in the points X and Y, points of the required line of intersection. Other points on the line of intersection are found in the same way. A curved line joining in the proper order the points thus found is the required line of intersection of the cylinders.

For construction purposes, it is much better to have a few well chosen planes than many poorly chosen ones. For example: In Fig. 83, use planes which contain the extreme or bounding elements of the cylinders in the top view. These planes will locate the points in which the top view of the curve of intersection touches or is tangent to the top views of these extreme elements. At the points thus located, the top view of the curve of intersection changes from visible to invisible or from invisible to visible if it changes at any place. Some of these are also the points in which the extreme elements of the surfaces come into view. Sometimes the extreme elements do not come into view until they have passed beyond the view of the other surface. In like manner, points on the extreme elements of the surfaces of the front view should be located. These points have similar significance with respect to the front view.

To draw a tangent to the curve of intersection at any point. Select the point at which the tangent is to be drawn and pass a plane tangent to each cylinder at this point. The intersection of these planes is the required tangent to the curve.

154. In the preceding problem, the bases of both cylinders are taken in one plane. If the bases are not both in one plane, the elements of either cylinder can be extended until they cut

the plane of the base of the other cylinder, thus locating a new base. The problem can then be solved as before. Another method of solution in case the bases of both cylinders do not lie in one plane is to find the lines of intersection of the cutting planes with the planes of the bases of both cylinders. From the points where these lines cross the curves of the bases, draw elements of the cylinders which will intersect in points of the required curve of intersection.

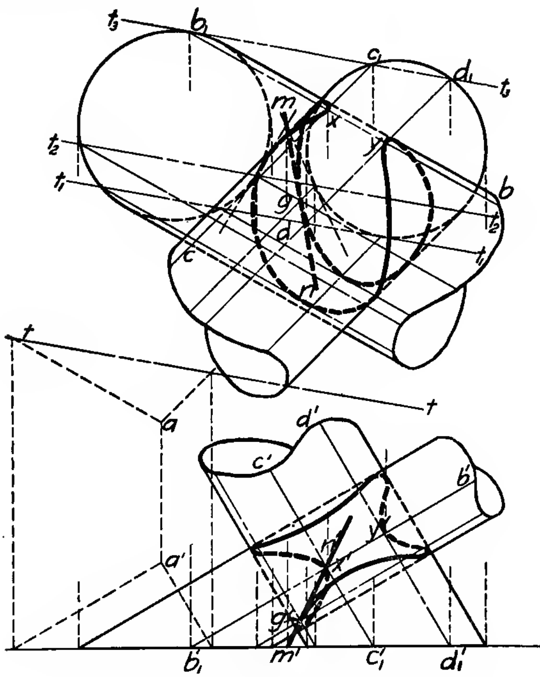


FIG. 85.—*Intersection of two cylinders.*

If the bases of both cylinders are circles in the same plane, as shown in Fig. 85, or if the bases are circles in parallel planes, points on the curve of intersection can be found by passing planes parallel to the plane of the bases. Each of these planes will cut circles from the cylinders and these circles will intersect

in points of the required curve of intersection.

In case a model of the cylinders as shown in Fig. 85 is desired, it is necessary to develop both surfaces by the method of Art. 143. The line of intersection of the surfaces should be laid off on the development of one cylinder, and if a free passage through both cylinders is desired, it must be laid off on the development of each cyclinder. With the developments as patterns, materials can be cut which will roll into the desired form.

155. To find the line of intersection of a cone and cylinder.

Let the surfaces be given as in Fig. 86.

Analysis. Pass a system of planes through the vertex of the cone and parallel with the elements of the cylinder. These planes will cut from each surface straight line elements which will intersect in points of the required line of intersection. A line through the vertex of the cone and parallel with the elements of the cylinder will lie in all of the planes.

Construction. The line V_1X, through the vertex of the cone and parallel with the elements of the cylinder, lies in all the planes and pierces the plane of the bases at X, a point common to all of the lines which these planes cut from the plane of the bases. Through x, draw the lines xt, xt_1, etc., cutting the bases of both surfaces. From the points where these lines cross the bases draw elements of the surfaces. These elements will intersect in points of the required line of intersection. In this manner the curve of intersection as shown in Fig. 86 is found.

The visible part of the curve is determined and the tangent is drawn in the same manner as in Arts 151 and 153. If the bases of the surfaces are not in the same plane, the methods suggested in Art. 154 will solve the problem.

As with the cylinders, the problem can be solved with a series of planes parallel with the plane of the bases, providing the bases of both surface are circles lying in one plane.

156. To determine in advance the nature of the curves in which cylinders and cones intersect. In the system of planes used in Fig. 86, there are two T and T_2, which are tan-

gent to the cone. xt and xt_2 are the intersections of these planes with the plane of the bases of the surfaces. The plane T is tangent to the cone on the back side and the plane T_2 is tangent to the cone on the front side. Both of these tangent planes cut the cylinder. It is evident then that the cylinder extends

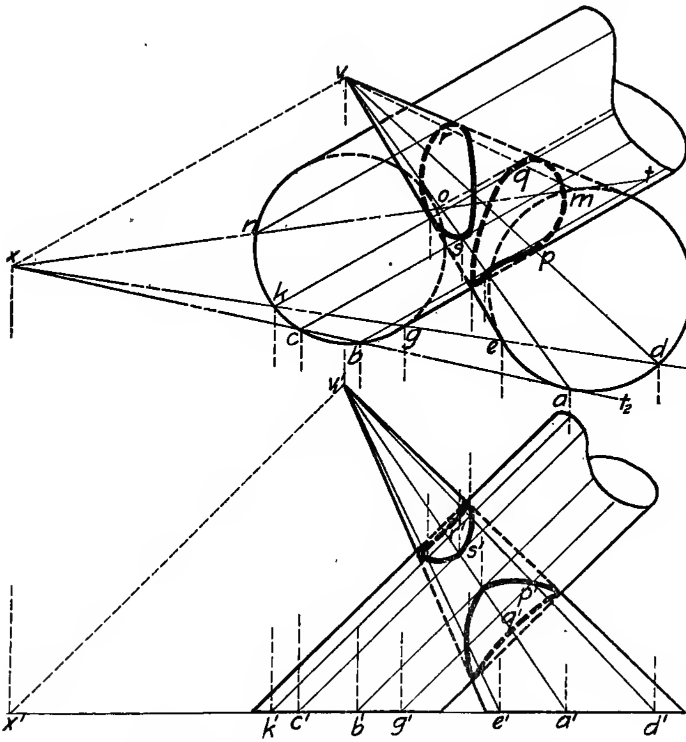


FIG. 86.—*Intersection of cone and cylinder.*

beyond the cone on both sides; the cone running into the cylinder from one side and out on the other side, forming two distinct curves of intersection, as shown in the figure. If one of these tangent planes had cut the base of the cylinder while the other did not, there would have been one continuous curve of intersection. If the cylinder had been between these planes

and not cut by either of them, there would again have been two distinct curves of intersection, where the cylinder runs into the cone from one side and where it runs out on the other. In the case where the planes are tangent to both cone and cylinder, the two curves of intersection just touch each other.

The same method may be used to determine the nature of the curve of intersection of two cylinders or of two cones.

157. To find the line of intersection of two cones.

Analysis. Pass a system of planes through the vertices of both cones. These planes will cut straight lines from each cone which will intersect in points of the required line of intersection. A line joining the vertices of the cones will lie in all of these planes and will pierce the plane of the bases in a point common to all the lines which these planes cut from the plane of the bases.

Let the construction be made for finding the curve of intersection. Also determine which part of the curve is visible and draw a tangent to the curve by methods previously given.

In what other way can the cutting planes be passed to find the curve of intersection?

158. To find the line of intersection of a sphere and a cone or a sphere and a cylinder.

Analysis 1. If the cone or cylinder has a circle for a base, the planes can be passed parallel to the base. Each plane will cut a circle from the sphere and a circle from the cone or cylinder. The intersections of these circles are points on the required line of intersection.

Analysis 2. Take a series of planes which contain the rectilinear elements of the cone or cylinder and are perpendicular to H or V. These planes will cut circles from the sphere which will intersect the elements cut from the cone or cylinder in points of the required line of intersection. To find these points, it may be necessary to revolve the planes until they are parallel to one of the planes of projection. In the case of the cylinder, an auxiliary view which shows the circles cut from the sphere in their true sizes gives a satisfactory solution.

Let the construction be made in accordance with the above analysis.

To draw a straight line tangent to the curve of intersection at any point. The required tangent will be the line of intersection of two planes, one tangent to the cone or cylinder and the other tangent to the sphere at the given point.

159. To find the line of intersection of a torus and cylinder when their axes are perpendicular to each other.

Analysis. Pass a system of planes which will cut both surfaces and which are perpendicular to the axis of the torus. Each of these planes will cut circles from the torus and straight lines from the cylinder. The intersections of these circles with the straight lines are points on the required line of intersection of the surfaces.

Construction. Take an auxiliary view of both surfaces by looking in the direction of the axis of the cylinder. In this view, the cutting planes will appear as straight lines. From the auxiliary view of the points where these lines cross the curves representing the surfaces, projecting lines are drawn to the view of the surfaces taken by looking in the direction of the axis of the torus. This view will show the points in which the circles cut from the torus intersect the lines cut from the cylinder. The line joining in proper order the points thus found is the required line of intersection of the torus and cylinder.

160. General method for finding the line of intersection of two surfaces A and B. Pass a system of surfaces, usually planes, in such a way that they will cut both of the given surfaces. Each of the third surfaces will cut a line from surface A which will intersect the line which it cuts from surface B in points of the line of intersection of A and B. The third surfaces should be passed in such a direction that they will cut the simplest lines possible from the surfaces A and B, and these lines should be in simple position with respect to the planes of projection. If possible cut either straight lines or circles from the given surfaces for these are the simplest lines to draw with a straight edge and compass.

PROBLEMS

161. Each of the following problems can be solved in a 10″ x 14″ rectangle with the base line through the center of the rectangle parallel to its longer side.

1. Find the intersection of an oblique cone and a right circular cylinder. The bases of both surfaces are circles on a horizontal plane.

2. Find the intersection of a right circular cone with a sphere. The base of the cone is on a horizontal plane and the center of the sphere is not on the axis of the cone.

3. Find the intersection of a right circular cylinder with a right circular cone. The axes of both surfaces are perpendicular to H but do not coincide.

4. Find the intersection of a sphere with a right circular cylinder. The axis of the cylinder is parallel to H and 30° to V.

5. Find the intersection of a right circular cone with a right circular cylinder. The base of the cone is on a horizontal plane and the axis of the cylinder is parallel to G. L.

6. Find the intersection of a triangular prism with a sphere. The axis of the prism is perpendicular to V.

7. Find the line of intersection of a cone with a triangular prism.

8. Find the line of intersection of a torus and a right circular cylinder. The axes of the surfaces are at right angles to each other but do not intersect.

9. Find the line of intersection of the two given oblique cylinders, which have their bases in a horizontal plane, and draw a tangent to the curve at some point of the curve not on an extreme element of either surface.

$$\text{Cylinder A}\begin{cases}\text{Center of lower base, } A(3\tfrac{1}{2},\ X,\ -4\tfrac{3}{4})\\ \text{Center of upper base, } O(-3\tfrac{1}{2},\ \tfrac{7}{8},\ -\tfrac{3}{4})\\ \text{Radius of base } 1\tfrac{3}{8}''\end{cases}$$

$$\text{Cylinder B}\begin{cases}\text{Center of lower base, } B(-3\tfrac{1}{2},\ Y,\ -4\tfrac{3}{4})\\ \text{Center of upper base, } C(\tfrac{1}{2},\ 3\tfrac{1}{2},\ -\tfrac{3}{4})\\ \text{Radius of base } 1\tfrac{3}{4}''\end{cases}$$

NOTE.—X may vary from $1\frac{3}{4}''$ to $2\frac{3}{4}''$ and Y may vary from $1\frac{1}{4}''$ to $1\frac{7}{8}'$ as the instructor assigns.

10. Find the line of intersection of the given oblique cylinder and the given oblique cone, which have their bases in a horizontal plane, and draw a tangent to the curve at some point of the curve not on an extreme element of either surface.

$$\text{Cylinder A}\begin{cases}\text{Center of the lower base, } A(\tfrac{1}{4}, X, -4\tfrac{3}{4})\\ \text{Center of the upper base, } C(-3, 1\tfrac{3}{4}, -1\tfrac{1}{4})\\ \text{Radius of the base, } 1\tfrac{1}{2}''\end{cases}$$

$$\text{Cone B}\begin{cases}\text{Center of the base, } B(-3\tfrac{1}{4}, Y, -4\tfrac{3}{4})\\ \text{Vertex, } O(1\tfrac{1}{2}, 4\tfrac{1}{4}, 0)\\ \text{Radius of the base, } 1\tfrac{3}{4}''\end{cases}$$

NOTE.—X may vary from $1\frac{1}{2}''$ to $2\frac{5}{8}''$ and Y may vary from 1″ to 2″ as the instructor assigns.

11. Find the line of intersection of two oblique cones which have their bases in a horizontal plane, and draw a line tangent to the curve of intersection at some point of the curve not on an extreme element of either surface.

$$\text{Cone A}\begin{cases}\text{Center of the base, } A(4\tfrac{1}{4}, X, -4\tfrac{3}{4})\\ \text{Vertex of the cone, } O(1, 3\tfrac{1}{4}, -2)\\ \text{Radius of the base, } 1\tfrac{1}{2}''\end{cases}$$

$$\text{Cone B}\begin{cases}\text{Center of the base, } B(0, Y, -4\tfrac{3}{4})\\ \text{Vertex of the cone, } C(6\tfrac{1}{2}, 4\tfrac{1}{4}, 0)\\ \text{Radius of the base, } 1\tfrac{3}{4}''\end{cases}$$

NOTE.—X may vary from $1\frac{3}{4}''$ to $3\frac{1}{4}''$ and Y may vary from 2″ to 3″ as the instructor assigns.

12. Find the line of intersection of the two given cones, which have their bases in a horizontal plane, and draw a line tangent to the curve of intersection at some point of the curve not on an extreme element of either surface.

$$\text{Cone A}\begin{cases}\text{Center of the base, } A(0, 3\tfrac{1}{2}, -6\tfrac{3}{4})\\ \text{Vertex of the cone, } V(0, 3\tfrac{1}{2}, 0)\\ \text{Radius of the base, } 3''\end{cases}$$

$$\text{Cone B}\begin{cases}\text{Center of the base, } B(\tfrac{1}{4}, 4, -6\tfrac{3}{4})\\ \text{Vertex of the cone, } V(-3\tfrac{1}{4}, 3, -1\tfrac{1}{4})\\ \text{Radius of the base, } 2\tfrac{7}{8}''\end{cases}$$

NOTE.—Take G. L. through the center of the sheet and parallel to the shorter side.

13. Find the intersection of a right circular cone with a right circular cylinder. The axis of the cylinder is perpendicular to H; the axis of the cone is parallel to H and makes 30° with V. Radius of base 1″. Develop the surface of one-fourth of the cone or cylinder showing the curve of intersection on the development.

14. Find the line of intersection of a right circular cone with a right circular cylinder. The axis of the cone is $A(-4, 1\frac{3}{8}, -4)$ $B(-4, 1\frac{3}{8}, -1)$, radius of base $1\frac{1}{2}''$. The axis of the cylinder is $M(-6, 1\frac{3}{16}, -3)$ $N(-1\frac{1}{2}, 1\frac{3}{16}, -3)$, radius of base 1″. Develop the surface of one-fourth of the cone or cylinder showing the curve of intersection on the development.

15. Find the intersection of a torus and a right circular cylinder. The axis of the torus is perpendicular to H, through the point $(0, 1\frac{3}{8}, -2)$, radii of circles representing the surface in the top view $\frac{7}{8}''$ and $3\frac{3}{8}''$ respectively, centers of circles representing the torus in the front view $3\frac{1}{2}''$ below the center line. The axis of the cylinder is $A(-1\frac{13}{16}, -2, -3\frac{13}{16})$ $B(-1\frac{13}{16}, 5, -3\frac{13}{16})$, radius of base $1\frac{1}{4}''$. Develop one-fourth of the cylinder, showing the curve of intersection on the development. (NOTE. Take base line parallel to the shorter side of sheet for this problem).

16. Find the line of intersection of a sphere with a right circular cylinder. The center of the sphere is at $C(-4, 2\frac{1}{2}, -2\frac{1}{2})$, radius 2". The axis of the cylinder is parallel to H and 30° with V, passing through the point $A(-4, 2, -1\frac{1}{2})$, radius of base 1''. The axis of the cylinder runs backward to the left. Develop one-fourth of the cylinder, showing the curve of intersection on the development.

17. Find the line of intersection of a right circular cone and a sphere. The center of the base of the cone is $(0, 4\frac{1}{4}, -3\frac{3}{4})$, radius of base $1\frac{3}{4}''$, altitude 5". Center of sphere $(-1\frac{1}{8}, 3\frac{1}{2}, -1\frac{3}{4})$, radius $1\frac{1}{2}''$ Develop one-fourth of the cone, show the curve of intersection on the development. (NOTE. Take base line parallel to shorter side of sheet for this problem).

18. Find the line of intersection of a cylinder and a sphere. The center of the base of the cylinder is at $A(5, 2, -4\frac{1}{2})$ and the radius is $1\frac{3}{4}'$. The axis makes 45° with H, is parallel with V, and slopes upward to the left. The center of the sphere is at $O(3, 2\frac{3}{8}, -2\frac{1}{4})$, radius 2". Develop one-quarter of the cylinder, showing the curve of intersection on the development.

19. Find the line of intersection of an oblique cylinder with a right circular cone. The bases of both surfaces are circles in the same horizontal plane. The center of the base of the cylinder is $A(-\frac{3}{4}, 2\frac{3}{8}, -4)$, radius $1\frac{3}{8}''$, the elements are parallel to V, and 30° to H, sloping upward to the left. The center of the base of the cone is $B(-4\frac{1}{4}, 2\frac{3}{4}, -4)$, radius 2", altitude $4\frac{1}{2}''$. Develop one-fourth of the cone, showing the curve of intersection on the development.

20. $A(-6, 1\frac{3}{4}, -1\frac{3}{4})$ $B(-2\frac{1}{2}, 1\frac{3}{4}, -1\frac{3}{4})$ is the axis of a right square prism. The plane of one face of the prism makes 30° with H. side of base $1\frac{1}{2}''$. Find the intersection of this prism with a sphere with center at $C(-4\frac{1}{4}, 1\frac{9}{16}, -1\frac{9}{16})$, radius 1". Develop two faces of the prism, showing the curve of intersection on the development.

21. Find the line of intersection of a sphere and a right circular cylinder. The axis of the cylinder is $A(-6, 3\frac{1}{2}, -3\frac{1}{4})$ $B(-2\frac{1}{2}, 0, -3\frac{1}{4})$, radius of base $\frac{7}{8}''$. The center of the sphere is $O(-4\frac{1}{2}, 2\frac{3}{4}, -2\frac{3}{4})$, radius $1\frac{1}{2}''$. Develop one-fourth of the cylinder, showing the curve of intersection on the development.

22. Find the line of intersection of two cones which have their circular bases in the same horizontal plane. The center of the base of one cone is at A$(1\frac{1}{4}, 2\frac{1}{2}, -4\frac{1}{2})$, vertex at O$(-6\frac{1}{2}, 2\frac{1}{2}, \frac{1}{2})$, radius of base $1\frac{3}{4}''$. The center of the base of the other cone is at B$(-2\frac{1}{2}, 2, -4\frac{1}{2})$, vertex at C$(--2\frac{1}{2}, 2, -1)$, radius of base $1\frac{3}{4}''$.

23. Find the intersection of a right circular cylinder with a right circular cone. The vertex of the base of the cone is at C$(-4\frac{1}{2}, 3\frac{1}{4}, -4\frac{1}{2})$, vertex V$(-4\frac{1}{2}, 3\frac{1}{4}, 0)$, radius of base 2″. The center of the base of the cylinder is O$(-2, 2, -3)$, axis parallel to H, 30° to V, sloping backward to the left; radius of base of cylinder $1\frac{1}{4}''$. Do not draw the front view of the bases of the cylinder unless they are used in getting the curve of intersection. Develop one-quarter of the cone or cylinder, showing the curve of intersection on the development.

24. Find the intersection of two right circular cylinders. The axes are A$(-4, -1, -3\frac{1}{4})$ B$(-4, 4\frac{3}{4}, -3\frac{1}{4})$ and C$(-1\frac{3}{4}, -\frac{1}{4}, -2\frac{7}{8})$ E$(-6, 4, -2\frac{7}{8})$, and the diameters are $2\frac{1}{8}''$ and $1\frac{3}{4}''$ respectively. Develop the surface of one cylinder showing the curve of intersection on the development.

25. Find the intersection of a right circular cylinder and a right circular cone. The base of the cylinder is a $2\frac{3}{4}''$ horizontal circle with center at A$(\frac{3}{4}, 1\frac{5}{8}, -4\frac{3}{4})$. The axis of the cylinder is parallel to V, makes 30° with H, and slopes upward to the left. B$(-2\frac{3}{4}, 2, -\frac{1}{4})$ C$(-2\frac{3}{4}, 2, -4\frac{3}{4})$ is the axis of the cone whose base is a horizontal circle 4″ in diameter with center at C.

26. A$(2\frac{1}{8}, 1, -3\frac{1}{4})$ B$(5, 3\frac{7}{8}, -3\frac{1}{4})$ is the axis of a right circular cylinder which is $2\frac{1}{4}''$ in diameter. A right circular cone with axis C$(4, 2\frac{1}{2}, -\frac{1}{4})$ E$(4, 2\frac{1}{2}, -4\frac{3}{4})$ has a $3\frac{1}{8}''$ horizontal circle with center at E as a base. Find the intersections of the cylinder and cone and devolop the surface of the cylinder.